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BANACH *-ALGEBRA VALUED INNER PRODUCTS

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1. Introduction

W.L. Paschke [2] investigated right modules over B^*-algebra B which possess a B-valued inner product respecting the module mapping. J.G. Bennett [1] studied properties of vector spaces equipped with B^*-algebra valued inner products. In this paper, we will be obtained some properties of vector spaces equipped with Banach *-algebra valued inner products.

2. Notation and Preliminaries

A Banach algebra A with an involution $*$ satisfying $\|a^*\|=\|a\|$ for all $a\in A$ will be called a *Banach *-algebra*. Throughout this paper, A denotes a Banach *-algebra with a multiple identity e and X denotes a complex linear space. A is *reduced* if $\{a\in A \mid f(a^*a)=0$ for all positive functionals f on $A\}=\{0\}$. B^*-algebras are reduced. A is *symmetric* if the inverse $(e+a^*a)^{-1}$ exists in A for all $a\in A$. A is symmetric if and only if $-a^*a$ is quasi-regular in A for all $a\in A$. If A is reduced, then A is symmetric ([3] Corollary, p266, and II Proposition, p. 303). All algebras and linear spaces are those over the complex field $\mathbb{C}$. It is assumed that all algebras possess a multiple identity e.

3. A-valued inner product spaces

DEFINITION 3.1. X will be called an *A-valued inner product space* if it is equipped with a map $\langle\cdot,\cdot\rangle : X\times X\longrightarrow A$ such that

(i) $\langle\cdot,\cdot\rangle$ is linear in the first entry,

(ii) $\langle x,x\rangle\geq 0$, i.e. $\langle x,x\rangle$ is a positive element of A, for all $x\in X$,

(iii) $\langle x,y\rangle=\langle y,x\rangle^*$, for all $x,y\in X$.

The map $\langle\cdot,\cdot\rangle$ is an *A-valued inner product* on X.

Suppose J is a *-subalgebra of a symmetric Banach *-algebra A and define $\langle x,y\rangle=y^*x$ for $x,y\in J$. Then J is an A-valued inner product space. Suppose J is a *-subalgebra of a *-algebra A, B is a Banach *-algebra, and $\Phi : A\longrightarrow B$ is a positive operator (i.e. Φ is linear and $\Phi(a^*a)\geq 0$, for all $a\in A$) with $\Phi(a^*)$

$=\Phi(a)^*$, for all $a\in A$. Define $<a,b>=\Phi(b^*a)$ for a, $b\in J$. Then J is a B-valued inner product space. Otherwheres, examples of such objects are numerous. A-valued inner product $<\cdot,\cdot>$ is a conjugate linear in the second entry. The conditions of A-valued inner product space are independent of each other.

An A-valued inner product space X will be called definite if $<x,x>=0$ only if $x=0$. In general, A-valued inner product spaces are not necessarily definite. Hence we may consider the following proposition.

PROPOSITION 3.2. *Suppose A is reduced and X is an A-valued inner product space. Let N be the set of elements x in X such that $<x, x>=0$. Then N is the linear subspace of X, and the quotient space X/N forms a definite A-valued inner product space.*

PROOF. Let f be a positive functional on A. A map $(x,y)\longrightarrow f(<x, y>)$ defines a pseudo-inner product on X. By Cauchy Schwarz-inequality, $f(<x+y, x+y>)=0$, for $x,y\in N$. Since A is reduced, $<x+y, x+y>=0$, for x, $y\in N$. Hence N is a linear subspace of X. Define $<x+N, y+N>=<x, y>$, for x, $y\in N$. Then $<\cdot,\cdot>$ is well-defined, and it is a definite A-valued inner product.

Let X be a complex linear space $\mathbb{C}^2$ with coordinate vector operations. Let A be the algebra of all matrices of the form

$$\begin{pmatrix}\alpha, & \beta\\ 0, & \alpha\end{pmatrix}$$

with $\alpha\in\mathbb{C}$, $\beta\in\mathbb{C}$. Define

$$\left\|\begin{pmatrix}\alpha, & \beta\\ 0, & \alpha\end{pmatrix}\right\|=|\alpha|+|\beta| \text{ and } \begin{pmatrix}\alpha, & \beta\\ 0, & \alpha\end{pmatrix}^*=\begin{pmatrix}\bar{\alpha}, & \bar{\beta}\\ 0, & \bar{\alpha}\end{pmatrix}.$$

Then it is well-known that A is a Banach $*$-algebra. It is easy to show that A is not reduced. Define

$$<(\alpha_1, \beta_1), (\alpha_2, \beta_2)>=\begin{pmatrix}0, & \alpha_1\bar{\beta}_2+\beta_1\bar{\alpha}_2\\ 0, & 0\end{pmatrix}.$$

Then X is an A-valued inner product space. We can take x, $y\in N$ with $x+y\notin N$. Thus the hypothesis in 3.2. Proposition, that A is reduced, is essential.

Suppose X is an A-valued inner product space. Let P_A be the set of all positive functionals f on A such that $f(e)\leq 1$. Define

$$\|x\|_X=\sup f(<x,x>)^{\frac{1}{2}},\ f\in P_A.$$

For $f\in P_A$, $x\longrightarrow f(<x, x>)^{\frac{1}{2}}$ is a semi-norm on X. Hence $\|\cdot\|_X$ is a semi-norm on X. When A is reduced and $<\cdot,\cdot>$ is definite, $\|\cdot\|_X$ is a norm on X.

Define $\|x\|_*=\|<x,x>\|^{\frac{1}{2}}$. Then $\|\cdot\|_*$ is not always a semi-norm on X. If A is a symmetric Banach *-algebra, then $\|x\|_X\leq\|x\|_*$. If A is a B^*-algebra, then $\|x\|_X=\|x\|_*$.

PROPOSITION 3.3. *Suppose A is a symmetric Banach *-algebra, and X is an A-valued inner product space.*

(i) *Assume that $\|a\|^2=\|a^2\|$, for all $a\in A$. Then $\|<x, y>\|\leq\|x\|_X\|y\|_X$, for all $x,y\in X$. In particular, $\|x\|_X=\|x\|_*$, for all $x\in X$.*

(ii) *Assume that A is reduced. Then $<x, y>\geq<y, x>$ if and only if $f(<x, y>)\leq\|x\|_X\|y\|_X$, for all $f\in P_A$.*

PROOF. (i)Let x, y be elements of X. By Schwarz-inequality, $|f(<x, y>)|^2\leq f(<x, x>)\, f(<y, y>)\leq\|x\|_X^2\|y\|_X^2$, for all $f\in P_A$. Hence $|f(<x, y>)|\leq\|x\|_X\|y\|_X$, for all $f\in P_A$. Since A is commutative ([6], p262) by ([3] Corollary 4, p308), $\lim\|<x, y>^n\|^{1/n}=\sup f(<x, y>)$, where the supremum is taken over all indecomposable normalized positive functionals f. By the hypothesis, $\|<x, y>\|=\lim\|<x, y>^{2^n}\|^{1/2^n}=\lim\|<x, y>^n\|^{1/n}\leq\|x\|_X\|y\|_X$.

(ii) The proof is obvious.

REMARKS 3.4. (1) 3.3 Proposition (i) is not true if A fails to be the condition that $\|x\|^2=\|x^2\|$, for all $x\in A$.

(2) 3.3 Proposition (ii) is not true if the condition that A is reduced, is omitted.

4. A-valued inner product modules

DEFINITION. 4.1. An A-valued inner product space X will be called an *A-valued inner product module* if it is a right A-module satisfying

$$<xa, y>=<x, y>a, \text{ for all } x, y\in X, a\in A.$$

W. L. Paschke [2] proved the following propositions for the B^*-algebra valued inner product modules (pre-Hilbert B-modules). In this section, we consider them in A-valued inner product modules X.

PROPOSITION. 4.2. *Suppose A is a commutative Banach *-algebra and X is an A-valued inner product module. Then $\|xa\|_X\leq\|x\|_X\|a\|$, for all $x\in X$, $a\in A$.*

PROOF. Let a be an element of A. For $f\in P_A$, let φ be a positive functional on A defined by $\varphi(z)=f(a^*za)$. Since $\varphi(b^*b)\leq\|b^*b\|\varphi(e)$ for all $b\in A$, $f(a^*b^*ba)\leq\|b^*b\|f(a^*a)$, for all $b\in A$. Since A is commutative, it is easy to show that $\|xa\|_X\leq\|a\|\|x\|_X$, for all $x\in X$, $a\in A$.

Let B be a symmetric Banach $*$-algebra. Then $\|b^*b\|e-b^*b\geq 0$, for all $b\in B$. By ([1], p6), we have

PROPOSITION 4.3. *Suppose B is a symmetric Banach $*$-algebra, and X is an A-valued inner product space. Then a left-module action of B on X satisfying $\langle bx, y\rangle=\langle x, b^*y\rangle$ for all x, $y\in X$, $b\in B$ has properties that*

(i) $\langle bx, bx\rangle\leq\|b\|^2\langle x, x\rangle$, *for all $x\in X$, $b\in B$, and*

(ii) $\|bx\|_X\leq\|b\|\|x\|_X$, *for all $x\in X$, $b\in B$.*

PROPOSITION. 4.4. *Suppose A is a reduced Banach $*$-algebra and B is a closed $*$-subalgebra of A. If $T: B\longrightarrow A$ is a linear map such that for some $K\geq 0$, we have $T(x)^*T(x)\leq Kx^*x$, for all $x\in B$. Then $T(x)=T(e)x$, for all $x\in B$.*

PROOF. Define $T_0: B\longrightarrow A$ by $T_0(x)=\{1/2(K+\|T(e)\|^2)\}^{\frac{1}{2}}(T(x)-T(e)x)$. Then $T_0(e)=0$. Easy computations show that $T_0(x)^*T_0(x)\leq x^*x$. Define a pseudo-norm on A as following

$$|x|=\sup f(x^*x),\ f\in P_A.$$

Since A is reduced, A is an A^*-algebra with $|x|$ as an auxiliary norm ([4], Corollary 4.6.10, p226). Let $\overline{A}$ be a completion of A by $|\cdot|$. Then $\overline{A}$ is a B^*-algebra. Let $\overline{B}$ be a closure of B in $\overline{A}$. Then $\overline{B}$ is a closed $*$-subalgebra of $\overline{A}$. Since $|T(x)|\leq K|x|$, T is a bounded operator w.r.t. $|\cdot|$ and T_0 is a bounded operator w.r.t. $|\cdot|$. Indeed, there exists $k>0$ such that $|T_0(x)|\leq k|x|$, for all $x\in B$. By ([5], 17 Theorem, p23), there exists an extension bounded operator $\overline{T}_0:\overline{B}\longrightarrow\overline{A}$ of T_0. Since $\overline{T}_0$ is continuous, $\overline{T}_0(x)^*\overline{T}_0(x)\leq x^*x$, for all $x\in\overline{B}$. By ([2], 2.7 Propostion), $\overline{T}_0(x)=\overline{T}_0(e)x$, for all $x\in\overline{B}$. Since $T_0(e)=0$, $T(x)=T(e)x$, for all $x\in B$.

LEMMA 4.5. *Suppose A is a symmetric and X is an A-valued inner product module. Then $\langle y, x\rangle\langle x, y\rangle\leq\|\langle y, y\rangle\|\langle x,x\rangle$, for all $x,y\in X$.*

Suppose A is a reduced Banach $*$-algebra, B is a closed $*$-subalgebra of A, $(X, \langle\cdot,\cdot\rangle_B)$, is a B-valued inner product module, and $(Y, \langle\cdot,\cdot\rangle_A)$, is a definite A-valued inner product module. Then, by 4.4 Proposition, 4.5 Lemma, and ([2], 2.8 Proposition), we have

PROPOSITION 4.6. *If $T: X \longrightarrow Y$ is a linear map such that $\langle Tx, Tx\rangle_A \leq \langle x, x\rangle_B$, for all $x \in X$. Then T is a bounded module map.*

Suppose A is reduced, B is a closed *-subalgebra of A, $(X, \langle\cdot,\cdot\rangle_B)$, is a B-valued inner product module, $(Y, \langle\cdot,\cdot\rangle_A)$, is an A-valued inner product module, and $T: X \longrightarrow Y$ is a linear map such that $\langle Tx, Tx\rangle_A \leq \langle x, x\rangle_B$, for all $x \in X$. Assume that there exists a nonzero element ξ of X such that $\langle T\xi, T\xi\rangle_A = \langle \xi, \xi\rangle_B$. Then we may be claimed that $\langle Tx, T\xi\rangle_A = \langle x, \xi\rangle_B$, for all $x \in X$. Let x be an element of X and t be any real number. Then we have $t(\langle T\xi, Tx\rangle_A + \langle Tx, T\xi\rangle_A) - t(\langle \xi, x\rangle_B + \langle x, \xi\rangle_B) \leq \langle x, x\rangle_B - \langle Tx, Tx\rangle_A$. Since t is arbitrary, $\langle T\xi, Tx\rangle_A + \langle Tx, T\xi\rangle_A = \langle \xi, x\rangle_B + \langle x, \xi\rangle_B$. It follows that $\langle Tx, T\xi\rangle_A = \langle x, \xi\rangle_B$. Let f be an element of P_B. Then X/N_f is a pre-Hilbert space with an inner product $(x+N_f, y+N_f) = f(\langle x, y\rangle_B)$, where $N_f = \{x \in X \mid f(\langle x, x\rangle_B) = 0\}$. Let $(H_f, \|\cdot\|_f)$ be a completion of X/N_f. Define $X+N_f \longrightarrow f(\langle x, \xi\rangle_B)$ is a well-defined linear functional on X/N_f. Since $|f(\langle x, \xi\rangle_B)| \leq \|x+N_f\|_f \, \|\xi+N_f\|_f$, there exists $T_{(f,\xi)} \in H_f$ such that $f(\langle x, \xi\rangle_B) = (x+N_f, T_{(f,\xi)})$ and $\|T_{(f,\xi)}\|_f \leq \|\xi+N_f\|_f$. Let $\hat{y}_f = y+N_f$. Suppose $g \in P_B$ such that $|g(x)| \leq |f(x)|$, for all $x \in B$. Since $N_f \subset N_g$, $x+N_f \longrightarrow x+N_g$ of X/N_f into X/N_g is a contractive map and extends a contractive map $V_{(f,g,\xi)}$ of H_f into H_g. For $x \in X$, let $V_{(f,g,\xi)}(\hat{x}_f) = \hat{x}_g$. Then we have the following proposition by ([2], 3.1 Proposition).

PROPOSITION 4.7 *Let A, B, X, Y, and T be as above. Assume that there exists a nonzero element ξ of X such that $\langle T\xi, T\xi\rangle_A = \langle \xi, \xi\rangle_B$ and $f, g \in P_B$ with $|g(x)| \leq |f(x)|$, for all $x \in X$. Then*

$$V_{(f,g,\xi)}(T_{(f,\xi)}) = T_{(g,\xi)}.$$

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