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MINIMAL SIMPLE EXTENSIONS OF TOPOLOGIES

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1. Introduction

In [2], the author introduced the concept of a simple extension $\mathscr{T}(A)$ of a topology $\mathscr{T}$ on a set X (see Definition 2.1).

A simple extension need not be a minimal extension, that is, there may exist a topology $\mathscr{U}$ on X for which $\mathscr{T}\subset\mathscr{U}\subset\mathscr{T}(A)$, $\mathscr{T}\neq\mathscr{U}\neq\mathscr{T}(A)$ (see Example 2.4). It is the purpose of this paper to study simple extensions of topology which are minimal.

In [2], the basic problem was to investigate the properties that are preserved under simple extensions, that is, if $(X,\mathscr{T})$ has a certain property, when will $(X,\mathscr{T}(A))$ have the same property?

In the present paper, we characterize minimal simple extensions (Theorem 3.1) and explore basically the same problem for such extensions.

2. Background

DEFINITION 2.1. Let $(X,\mathscr{T})$ be a space and $A\subset X$, $A\notin\mathscr{T}$. Then $\mathscr{T}(A)$ is the collection of sets of the form $O_1\cup(O_2\cap A)$, O_1 and O_2 in $\mathscr{T}$, and is called the *simple extension of* $\mathscr{T}$ *by* A (see [2]). We shall call $\mathscr{T}(A)$ a *minimal simple extension* if for each topology $\mathscr{U}$ for which $\mathscr{T}\subset\mathscr{U}\subset\mathscr{T}(A)$, then $\mathscr{T}=\mathscr{U}$ or $\mathscr{U}=\mathscr{T}(A)$. In this case, we write $\mathscr{T}\,\mathrm{imp}\,\mathscr{T}(A)$ ($\mathscr{T}$ immediately precedes $\mathscr{T}(A)$).

THEOREM 2.2. *Let* $(X,\mathscr{T})$ *be a space and* $A\subset X$, $A\notin\mathscr{T}$. *Then*

(1) $\mathscr{T}(A)$ *is a topology for* X

(2) $\mathscr{T}\subset\mathscr{T}(A)$ *and*

(3) $\mathscr{T}(A)=\sup\{\mathscr{T}, \{\phi, A, X\}\}$.

This is Theorem I.1.2 in [1].

THEOREM 2.3. *Let* $(X,\mathscr{T})$ *be a space and* $A\subset X$, $A\notin\mathscr{T}$. *If* $\mathscr{U}$ *is a topology for* X *and* $\mathscr{T}\subset\mathscr{U}\subset\mathscr{T}(A)$, *then* $\mathscr{U}=\mathscr{T}(A)$ *iff* $A\in\mathscr{U}$.

This is Lemma I.1.7 in [1].

EXAMPLE 2.4. Let $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi, \{a\}, X\}$. Let $B=\{b\}$. Then $\mathscr{T}(B)$ is a simple extension of $\mathscr{T}$, but $\mathscr{T}$ imp $\mathscr{T}(B)$ is false.

NOTATION. In a space $(X,\mathscr{T})$, B° denotes the interior of B, $c(B)$ the closure of B and $\mathscr{C}B$ the complement of B.

3. The Fundamental Theorem

THEOREM 3.1. *Let $(X,\mathscr{T})$ be a space and $A\subset X$, $A\notin\mathscr{T}$. Then $\mathscr{T}$ imp $\mathscr{T}(A)$ (see Definition 2.1) iff*

(1) $A-A^\circ$ is indiscrete and

(2) $O\in\mathscr{T}$, $O\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$ implies that $O-A^\circ$ and $A-A^\circ$ are separated.

PROOF. Necessity. Let $\mathscr{T}$ imp $\mathscr{T}(A)$; (1) we show that $A-A^\circ$ is indiscrete. It suffices to show that $O\in\mathscr{T}$, $O\cap(A-A^\circ)\neq\phi$ implies that $O\supset(A-A^\circ)$. Let then $b\in O\cap(A-A^\circ)$, $O\in\mathscr{T}$ and $a\in A-A^\circ$. Now $O\cap A\notin\mathscr{T}$ lest $b\in O\cap A\subset A^\circ$. Hence $\mathscr{T}\subset\mathscr{T}(O\cap A)\subset\mathscr{T}(A)$ and since $\mathscr{T}\neq\mathscr{T}(O\cap A)$, it follows that $\mathscr{T}(O\cap A)=\mathscr{T}(A)$. But $A\in\mathscr{T}(A)$ and therefore $A\in\mathscr{T}(O\cap A)$. Thus there exist O_1, O_2 in $\mathscr{T}$ such that $A=O_1\cup(O_2\cap(O\cap A))$. If $a\notin O$, then $a\in O_1\subset A^\circ$, a contradiction. Thus $A-A^\circ\subset O$. (2) Suppose $O\in\mathscr{T}$ and $O\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$. If $O\cap(A-A^\circ)\neq\phi$, then $O\supset(A-A^\circ)$ and $O\cup(A-A^\circ)=O\in\mathscr{T}$, a contradiction. Hence $O\cap(A-A^\circ)=\phi$ and $O\cap c(A-A^\circ)=\phi$. It follows then that $(O-A^\circ)\cap c(A-A^\circ)=\phi$. We show now that $(A-A^\circ)\cap c(O-A^\circ)=\phi$. Now $\mathscr{T}\subset\mathscr{T}(O\cup(A-A^\circ))\subset\mathscr{T}(A)$ and since $O\cup(A-A^\circ)\notin\mathscr{T}$, then $\mathscr{T}\neq\mathscr{T}(O\cup(A-A^\circ))$. It follows then that $\mathscr{T}(A)=\mathscr{T}(O\cup(A-A^\circ))$ and $A\in\mathscr{T}(O\cup(A-A^\circ))$. There exist then O_1 and O_2 in $\mathscr{T}$ for which $A=O_1\cup(O_2\cap(O\cup(A-A^\circ)))=O_1\cup(O_2\cap O)\cup(O_2\cap(A-A^\circ))$. Since $A\notin\mathscr{T}$, it follows that $O_2\cap(A-A^\circ)\neq\phi$ and by (1) above, $O_2\supset(A-A^\circ)$. It suffices to show that $O_2\cap(O-A^\circ)=\phi$. But $O_2\cap(O-A^\circ)\subset O_2\cap O\subset A^\circ$ and $O_2\cap(O-A^\circ)\subset\mathscr{C}A^\circ$. Thus $O_2\cap(O-A^\circ)=\phi$.

Sufficiency. Suppose (1) and (2) hold. We will show that $\mathscr{T}$ imp $\mathscr{T}(A)$. Suppose $\mathscr{T}\subset\mathscr{U}\subset\mathscr{T}(A)$ and $\mathscr{T}\neq\mathscr{U}$. By Theorem 2.3, it suffices to show that $A\in\mathscr{U}$. Let $U^*=O_1^*\cup(O_2^*\cap A)\in\mathscr{U}-\mathscr{T}$, $O_1^*\in\mathscr{T}$, $O_2^*\in\mathscr{T}$. Then $U^*=O_1^*\cup(O_2^*\cap A^\circ)\cup(O_2^*\cap(A-A^\circ))$ and hence $O_2^*\cap(A-A^\circ)\neq\phi$ lest $U^*\in\mathscr{T}$. By (1) $O_2^*\supset A-A^\circ$ and $U^*=O_1^*\cup(O_2^*\cap A^\circ)\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$. Let $O=O_1^*\cup(O_2^*\cap A^\circ)$; then $O\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$ and by (2), $(O-A^\circ)$ and $(A-A^\circ)$ are separated. Thus $(A-A^\circ)\cap c(O-A^\circ)=\phi$. Now $A^\circ\cup(U^*\cap\mathscr{C}c(O-A^\circ))\in\mathscr{U}$ and $A^\circ\cup(U^*\cap\mathscr{C}c(O-A^\circ))=A^\circ\cup((O\cup(A-A^\circ))\cap\mathscr{C}c(O-A^\circ))=A^\circ\cup(O\cap\mathscr{C}c(O-A^\circ))$

$\cup(A-A^\circ)=A^\circ\cup((O-A^\circ)\cap\mathcal{C}c(O-A^\circ))\cup((O\cap A^\circ)\cap\mathcal{C}c(O-A^\circ))\cup(A-A^\circ)=A^\circ$ $\cup(A-A^\circ)=A$. Hence $A\in\mathscr{U}$.

COROLLARY 3.2. *Let $(X,\mathscr{T})$ be a space and $A\subset X$, $A\notin\mathscr{T}$. Then $\mathscr{T}$ imp$\mathscr{T}(A)$ iff (a) $A-A^\circ$ is indiscrete and (b) $O\in\mathscr{T}$, $O\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$ implies that $(A-A^\circ)\cap c(O-A^\circ)=\phi$.*

PROOF. If $\mathscr{T}$imp$\mathscr{T}(A)$, then (a) holds by (1) of Theorem 3.1 and (b) holds by (2) of Theorem 3.1. Now let (a) and (b) hold. By Theorem 3.1, it suffices to show that $(O-A^\circ)\cap c(A-A^\circ)=\phi$ when $O\in\mathscr{T}$ and $O\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$. But by (a), $O\cap(A-A^\circ)=\phi$ and thus $O\cap c(A-A^\circ)=\phi$. Hence $(O-A^\circ)\cap c(A-A^\circ)=\phi$.

COROLLARY 3.3. *Let $(X,\mathscr{T})$ be a space and $A\subset X$, $A\notin\mathscr{T}$. If $\mathscr{T}$imp$\mathscr{T}(A)$, and $(X,\mathscr{T})$ is separable, then $(X,\mathscr{T}(A))$ is separable.*

PROOF. Let $\{x_i : i\geqq 1\}$ be dense in $(X,\mathscr{T})$ and take $y\in A-A^\circ$. Then $\{y\}\cup\{x_i : i\geqq 1\}$ is dense in $(X,\mathscr{T}(A))$. For let $\phi\neq O_1\cup(O_2\cap A)\in\mathscr{T}(A)$. Then $O_1\cup(O_2\cap A)=O_1\cup(O_2\cap A^\circ)\cup(O_2\cap(A-A^\circ))$.

Case 1 : $O_2\cap(A-A^\circ)=\phi$, Then $(O_1\cup(O_2\cap A^\circ))\cap\{x_i : i\geqq 1\}\neq\phi$.

Case 2 : $O_2\cap(A-A^\circ)\neq\phi$. Then by (1) of Theorem 3.1, $O_2\supset A-A^\circ$ and $y\in O_1\cup(O_2\cap A)$.

See Theorem 8 in [2] in this connection.

EXAMPLE 3.4. Let X be an infinite set and x^* a fixed element of X. If $\mathscr{T}=\{O : O\subset X$ and $x^*\notin O$ or $x^*\in O$ and $\mathcal{C}O$ is finite$\}$, then $\mathscr{T}$imp$\mathscr{T}(A)$ for no $A\subset X$.

PROOF. Let $x^*\in A$ and let $\mathcal{C}A$ be infinite. Then $\mathcal{C}A=B_1\cup B_2$ where $B_1\cap B_2=\phi$, B_1 and B_2 both being infinite. Let $O=A^\circ\cup B_1$. Then $O\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$, but $O-A^\circ$ and $A-A^\circ$ are not separated for $x^*\in(A-A^\circ)\cap c(O-A^\circ)$. (See (b) of Corollary 3.2.)

COROLLARY 3.5. *Let $(X,\mathscr{T})$ be a space and $A\notin\mathscr{T}$. If $\mathscr{T}$imp$\mathscr{T}(A)$, $C\subset A$, $C\cup\mathcal{C}A$ not closed, then $C\subset A^\circ$.*

PROOF. $A\cap\mathcal{C}C\notin\mathscr{T}$ and hence $(A^\circ\cap\mathcal{C}C)\cup((A-A^\circ)\cap\mathcal{C}C)\notin\mathscr{T}$. It follows then from (a) of Corollary 3.2 that $A-A^\circ\subset\mathcal{C}C$ and thus $(A-A^\circ)\cap C=\phi$. Hence $C\subset A^\circ$.

COROLLARY 3.6. *Let $(X,\mathscr{T})$ be a space and $A\notin\mathscr{T}$. For each $O\in\mathscr{T}$, suppose*

$A\subset O$ or $O\subset A$. Then $\mathscr{T}$ imp$\mathscr{T}(A)$.

PROOF. We employ Corollary 3.2.

(a) Suppose $O\cap(A-A^\circ)\neq\phi$. Then $O\not\subset A$ lest $O\subset A^\circ$. Thus $A\subset O$ and $A-A^\circ\subset O$. Hence $A-A^\circ$ is indiscrete.

(b) Suppose $O\in\mathscr{T}$ and $O\cup(A-A^\circ)\in\mathscr{T}(A)-\mathscr{T}$.

Case 1. $A\subset O$. Then $O\cup(A-A^\circ)=O\in\mathscr{T}$, a contradiction.

Case 2. $O\subset A$. Then $O\subset A^\circ$ and $O-A^\circ=\phi$. Thus $O-A^\circ$ and $A-A^\circ$ are separated.

Corollary 3.6 yields the following.

EXAMPLE 3.7. Let X be the reals and let $\mathscr{T}=\{O : O=\phi, O=X$ or $O=(-\infty, a)$ for some $a\in X\}$. Let $A=(-\infty,1]$. Then $\mathscr{T}$ imp$\mathscr{T}(A)$.

COROLLARY 3.8. *Let $(X,\mathscr{T})$ be a space with the following property: $O\in\mathscr{T}$ implies that $\mathscr{C}O\in\mathscr{T}$. If $A\notin\mathscr{T}$, then $\mathscr{T}$* imp*$\mathscr{T}(A)$ iff $A-A^\circ$ is indiscrete.*

PROOF. We employ Corollary 3.2. Let $O\in\mathscr{T}$ and $O\cup(A-A^\circ)\notin\mathscr{T}$. But $(A-A^\circ)\cap c(O-A^\circ)=(A-A^\circ)\cap(O-A^\circ)$ (since $O-A^\circ$ is closed). If $(A-A^\circ)\cap(O-A^\circ)\neq\phi$, then $O\cup(A-A^\circ)=O\in\mathscr{T}$ since $A-A^\circ$ is indiscrete.

COROLLARY 3.9. *Let $(X,\mathscr{T})$ be a space with the following property: $\phi\neq O\subset B\subset X$, $O\in\mathscr{T}$ implies that $B\in\mathscr{T}$. (For example, $\mathscr{T}$ is the cofinite, or cocountable topology.) If $\{x^*\}\notin\mathscr{T}$, then $\mathscr{T}$*imp*$\mathscr{T}(\{x^*\})$.*

PROOF. (a) $\{x^*\}-\{x^*\}^\circ=\{x^*\}$ and is indiscrete. (b) If $O\in\mathscr{T}$ and $O\cup\{x^*\}\notin\mathscr{T}$, then $O=\phi$ and $O-\{x^*\}^\circ$ and $\{x^*\}-\{x^*\}^\circ$ are separated.

COROLLARY 3.10. *Let $(X,\mathscr{T})$ be a space and $A\notin\mathscr{T}$. Then $\mathscr{T}(A)=\mathscr{T}\cup\{A\}$ iff* (1) *$O\in\mathscr{T}$, $O\cap(A-A^\circ)\neq\phi$ implies that $O\supset A$ and* (2) *$O\cap\mathscr{C}A\neq\phi$ implies that $O\cup A\in\mathscr{T}$.*

PROOF. Suppose that $\mathscr{T}(A)=\mathscr{T}\cup\{A\}$.

(1) Supoose that $O\in\mathscr{T}$ and $O\cap(A-A^\circ)\neq\phi$. Then by (a) of Corollary 3.2, $O\supset A-A^\circ$. But $O\cap A=(A-A^\circ)\cup(O\cap A^\circ)\in\mathscr{T}\cup\{A\}$. If $O\supset A^\circ$, then $O\supset A$. If $O\not\supset A^\circ$, then $O\cap A\neq A$ and hence $O\cap A\in\mathscr{T}$. Thus $(A-A^\circ)\cup(O\cap A^\circ)\in\mathscr{T}$ and $(A-A^\circ)\cup(O\cap A^\circ)\cup A^\circ=A\in\mathscr{T}$, a contradiction.

(2) Suppose $O\cap\mathscr{C}A\neq\phi$. Then $O\cup A\in\mathscr{T}(A)-\{A\}=\mathscr{T}$. Conversely, suppose that (1) and (2) hold. Let $O_1\cup(O_2\cap A)\in\mathscr{T}(A)-\mathscr{T}$. If $O_2\cap(A-A^\circ)=\phi$, then $O_1\cup(O_2\cap A)\in\mathscr{T}$, a contradition. Hence $O_2\supset A$ and thus $O_1\cup A\in\mathscr{T}(A)-\mathscr{T}$,

and $O_1 \cup A \notin \mathscr{T}$. By (2), $O_1 \cap \complement A = \phi$ and $O_1 \subset A$. Hence $O_1 \cup (O_2 \cap A) = A$.

COROLLARY 3.11. *Let $(X, \mathscr{T})$ be a space of the first category and suppose that $\mathscr{T}$ imp $\mathscr{T}(A)$. Then $\mathscr{T}(A)$ is of the first category iff $(A - A^\circ) \cap c(A^\circ) \neq \phi$.*

PROOF. Sufficiency. Let $X = \bigcup \{F_i : i \geqq 1\}$ where $\complement F_i \in \mathscr{T}$ for all i. Then F_i is closed in $(X, \mathscr{T}(A))$ for each i. Suppose further that the $\mathscr{T}$-int $F_i = \phi$ for each i. We will show that the $\mathscr{T}(A)$-int $F_i = \phi$ for each i. Suppose on the contrary that $\phi \neq O_1 \cup (O_2 \cap A) \subset F_i$ for some i. Then $O_2 \cap (A - A^\circ) \neq \phi$ and by (a) of Corollary 3.2, $O_2 \supset A - A^\circ$. Then $O_2 \cap A^\circ \neq \phi$ and F_i has a nonempty $\mathscr{T}$-int, a contradiction.

Necessity. Suppose that $(A - A^\circ) \cap c(A^\circ) = \phi$. Now $A \in \mathscr{T}(A)$ and $A - c(A^\circ) \in \mathscr{T}(A)$. It follows then that $A - A^\circ \in \mathscr{T}(A)$. It is clear that $A - A^\circ$ is indiscrete in $\mathscr{T}(A)$ as well as in $\mathscr{T}$. Suppose that $X = \bigcup \{F_i^* : i \geqq 1\}$, where $\complement F_i^* \in \mathscr{T}(A)$ for all i. It follows that $A - A^\circ \subset F_i^*$ for some i and hence the $\mathscr{T}(A)$-int$F_i^* \neq \phi$. Thus $(X, \mathscr{T}(A))$ is not of the first category.

LEMMA 3.12. *Let $(X, \mathscr{T})$ be a space and $x^* \in X$. Suppose $\{x^*\}$ is not closed, but $x^* \in O \in \mathscr{T}$ implies that $c(\{x^*\}) \subset O$. Then $\mathscr{T}$ imp $\mathscr{T}(\{x^*\} \cup \complement c(\{x^*\}))$ and $\mathscr{T}$ imp $\mathscr{T}(\complement\{x^*\})$.*

PROOF. See Corollary 5.4 and Corollary 6.4.

LEMMA 3.13. *Let $(X, \mathscr{T})$ be a first axiom Hausdorff space. Then $\mathscr{T}$ imp $\mathscr{T}(A)$ for no $A \subset X$.*

PROOF. Suppose on the contrary that $\mathscr{T}$ imp $\mathscr{T}(A)$ for some $A \notin \mathscr{T}$. Then $\complement A$ is not closed; take $a \in A \cap c(\complement A)$. Then there exists a sequence of distinct points $x_i \in \complement A$ such that $a = \lim x_i$. Let $E = \{a, x_2, x_4, x_6, \cdots\}$. Clearly E is compact and hence closed in $(X, \mathscr{T})$. Let $O = \complement E$. Then $O \cup (A - A^\circ) = O \cup A$ since $A^\circ \subset O$. But $O \cup A \in \mathscr{T}(A) - \mathscr{T}$ (if $O \cup A \in \mathscr{T}$, then x_i is eventually in $O \cup A$). Thus $O \cup (A - A^\circ) \in \mathscr{T}(A) - \mathscr{T}$, but $a \in (A - A^\circ) \cap c(O - A^\circ)$ ($a = \lim x_{2i+1}$). This contradicts (b) of Corollary 3.2.

See Theorem I.4.3 in [1].

THEOREM 3.14. *Let $(X, \mathscr{T})$ be a first axiom space and regular. Then $(X, \mathscr{T})$ is Hausdorff iff $\mathscr{T}$ imp $\mathscr{T}(A)$ for no $A \subset X$.*

PROOF. The necessity follows from Lemma 3.13.

Sufficiency. That $(X,\mathscr{T})$ is a T_1 space follows from Lemma 3.12. T_1 plus regular implies Hausdorff.

COROLLARY 3.15. *Let $(X,\mathscr{T})$ be metrizable. Then $\mathscr{T}$ imp$\mathscr{T}(A)$ for no $A\subset X$.*

4. $A^\circ=\phi$

THEOREM 4.1. *Let $(X,\mathscr{T})$ be a space and $A\notin\mathscr{T}$. If $A^\circ=\phi$, then $\mathscr{T}$ imp$\mathscr{T}(A)$ iff* (1) *A is indiscrete and* (2) *$O\cup A\notin\mathscr{T}$ implies that O and A are separated whenever $O\in\mathscr{T}$.*

PROOF. This follows from Theorem 3.1 and the fact that $O\cup A$ always is in $\mathscr{T}(A)$.

COROLLARY 4.2. *Let $(X,\mathscr{T})$ be a space and $\{x\}\notin\mathscr{T}$. Then $\mathscr{T}$ imp$\mathscr{T}(\{x\})$ iff $O\in\mathscr{T}$ and $O\cup\{x\}\notin\mathscr{T}$ implies that $x\notin c(O)$.*

COROLLARY 4.3. *Let $(X,\mathscr{T})$ be a space, $\phi\neq A\subset O^*$, $A\neq O^*\in\mathscr{T}$ and O^* minimal open. Then $\mathscr{T}$ imp$\mathscr{T}(A)$.*

PROOF. Firstly, $A\notin\mathscr{T}$ and $A^\circ=\phi$. We show that A is indiscrete. Suppose $O\in\mathscr{T}$ and $O\cap A\neq\phi$. Then $O\cap O^*\neq\phi$ and since O^* is minimal open, it follows that $O\supset O\cap O^*=O^*\supset A$.

Secondly, suppose $O\in\mathscr{T}$ and $O\cup A\notin\mathscr{T}$. Now $O\cap A=\phi$ lest $O\supset A$ and $O\cup A\in\mathscr{T}$. Hence $O\cap O^*=\phi$ and $A\cap c(O)=\phi$. It follows then that O and A are separated.

THEOREM 4.4. *Let $(X,\mathscr{T})$ be a space and $\mathscr{T}$ imp$\mathscr{T}(A)$. Suppose $B\subset X$ and $B^\circ\supset A^\circ$. Then $B\in\mathscr{T}(A)-\mathscr{T}$ iff $B-B^\circ=A-A^\circ$.*

PROOF. Necessity. ($B^\circ\supset A^\circ$ is not used in this part of the proof.) Let $B\in\mathscr{T}(A)-\mathscr{T}$. Then $\mathscr{T}\subset\mathscr{T}(B)\subset\mathscr{T}(A)$ and $\mathscr{T}\neq\mathscr{T}(B)$. Hence $\mathscr{T}(B)=\mathscr{T}(A)$ and $B\in\mathscr{T}(A)$, $A\in\mathscr{T}(B)$.

Thus $B=O_1\cup(O_2\cap A)$ for some $O_i\in\mathscr{T}$ and $B=O_1\cup(O_2\cap A^\circ)\cup(A-A^\circ)$. But $B=B^\circ\cup(B-B^\circ)$. It follows then that $B-B^\circ\subset A-A^\circ$.

Also, $A=O_1^*\cup(O_2^*\cap B)=O_1^*\cup(O_2^*\cap B^\circ)\cup(B-B^\circ)=A^\circ\cup(A-A^\circ)$. It follows that $A-A^\circ\subset B-B^\circ$ and hence $A-A^\circ=B-B^\circ$.

Sufficiency. Let $A-A^\circ=B-B^\circ$. Then $B-B^\circ\neq\phi$ and $B\notin\mathscr{T}$. We show that $B\in\mathscr{T}(A)$. Now $B=B^\circ\cup(B-B^\circ)=B^\circ\cup A^\circ\cup(A-A^\circ)=B^\circ\cup A\in\mathscr{T}(A)$.

COROLLARY 4.5. *Let $(X,\mathscr{T})$ be a space, $\mathscr{T}$ imp$\mathscr{T}(A)$ and $A^\circ=\phi$. Let $B\subset X$.*

Then $B \in \mathscr{T}(A) - \mathscr{T}$ *iff* $B - B^\circ = A$.

PROOF. This follows from Theorem 4.5 and the fact that $B^\circ \supset A^\circ$.

COROLLARY 4.6. *Let* $(X, \mathscr{T})$ *be a space and* $\mathscr{T}\,\text{imp}\,\mathscr{T}(A)$. *If* $A^\circ = \phi$, *then* A *is the smallest member of* $\mathscr{T}(A) - \mathscr{T}$.

PROOF. Let $B \in \mathscr{T}(A) - \mathscr{T}$. Then $B \supset B - B^\circ = A$ by Corollary 4.5.

5. A Indiscrete

THEOREM 5.1. *Let* $(X, \mathscr{T})$ *be a space,* $A \subset X$, $A \cup \mathscr{C}c(A) \notin \mathscr{T}$ *and* A *indiscrete. Then* $\mathscr{T}\,\text{imp}\,\mathscr{T}(A \cup \mathscr{C}c(A))$.

PROOF. Let $\mathscr{U}$ be a topology for X for which $\mathscr{T} \subset \mathscr{U} \subset \mathscr{T}(A \cup \mathscr{C}c(A))$, $\mathscr{T} \neq \mathscr{U}$. Let $U = O_1 \cup (O_2 \cap (A \cup \mathscr{C}c(A)) \in \mathscr{U} - \mathscr{T}$. Then $\mathscr{T} \subset \mathscr{T}(U) \subset \mathscr{U} \subset \mathscr{T}(A \cup \mathscr{C}c(A))$. By Theorem 2.3, it suffices to show that $A \cup \mathscr{C}c(A) \in \mathscr{T}(U)$. Now $O_2 \cap A \neq \phi$ lest $U \in \mathscr{T}$. Hence $O_2 \supset A$. $O_1 \cap A = \phi$ lest $U \in \mathscr{T}$. It follows then that $O_1 \cap c(A) = \phi$ and $O_1 \subset \mathscr{C}c(A)$. Thus $A \cup \mathscr{C}c(A) = \mathscr{C}c(A) \cup (O_2 \cap (O_1 \cup (O_2 \cap (A \cup \mathscr{C}c(A)))) = \mathscr{C}cA \cup (O_2 \cap U) \in \mathscr{T}(U)$.

COROLLARY 5.2. *Let* $(X, \mathscr{T})$ *be a space,* A *indiscrete and* $c(A) - A$ *not closed. Then* $\mathscr{T}\,\text{imp}\,\mathscr{T}(A \cup \mathscr{C}c(A))$.

PROOF. We need only show that $A \cup \mathscr{C}c(A) \notin \mathscr{T}$. But $\mathscr{C}(A \cup \mathscr{C}c(A)) = c(A) \cap \mathscr{C}A = c(A) - A$ which is not closed.

COROLLARY 5.3. *Let* $(X, \mathscr{T})$ *be a space and* $x^* \in X$. *If* $c(\{x^*\}) - \{x^*\}$ *is not closed, then* $\mathscr{T}\,\text{imp}\,\mathscr{T}(\{x^*\} \cup \mathscr{C}c(\{x^*\}))$.

PROOF. This follows from Corollary 5.2 and the fact that $\{x^*\}$ is indiscrete.

COROLLARY 5.4. *Let* $(X, \mathscr{T})$ *be a space and* $\{x^*\}$ *not closed. If* $x^* \in O \in \mathscr{T}$, *then* $c(\{x^*\}) \subset O$. *Then* $\mathscr{T}$ imp $(\{x^*\} \cup \mathscr{C}c(\{x^*\}))$.

PROOF. We use Theorem 5.1. If $\{x^*\} \cup \mathscr{C}c(\{x^*\}) \in \mathscr{T}$, then $c(\{x^*\}) = \{x^*\}$ and $\{x^*\}$ is closed.

COROLLARY 5.5. *Let* $(X, \mathscr{T})$ *be a space and* A *indiscrete. If* $A \notin \mathscr{T}$ *and is dense, then* $\mathscr{T}\,\text{imp}\,\mathscr{T}(A)$.

PROOF. $A = A \cup \mathscr{C}c(A) \notin \mathscr{T}$. By Theorem 5.1, $\mathscr{T}\,\text{imp}\,\mathscr{T}(A \cup \mathscr{C}c(A)) = \mathscr{T}(A)$.

6. $\mathscr{C}A$ Indiscrete

LEMMA 6.1. *Let $(X,\mathscr{T})$ be a space, $A\notin\mathscr{T}$, $\mathscr{C}A$ indiscrete. Then $\mathscr{T}$* imp*$\mathscr{T}(A)$ iff $\mathscr{C}A\subset O\in\mathscr{T}$, $O\cap A\notin\mathscr{T}$ implies that $X=A^\circ\cup O$.*

PROOF. Necessity. Let $\mathscr{C}A\subset O\in\mathscr{T}$ and $O\cap A\notin\mathscr{T}$. It suffices to show that $A-A^\circ\subset O$. By (a) of Corollary 3.2, it suffices to show that $(A-A^\circ)\cap O\neq\phi$. Now $O\cap A=(O\cap A^\circ)\cup(O\cap(A-A^\circ))\notin\mathscr{T}$. It follows then that $O\cap(A-A^\circ)\neq\phi$.

Sufficiency. Suppose $\mathscr{U}$ is a topology for X and $\mathscr{T}\subset\mathscr{U}\subset\mathscr{T}(A)$, $\mathscr{T}\neq\mathscr{U}$. We will show that $\mathscr{U}=\mathscr{T}(A)$. By Theorem 2.3, it suffices to show that $A\in\mathscr{U}$. Let $U^*\in\mathscr{U}-\mathscr{T}$. Then $U^*=O_1^*\cup(O_2^*\cap A)$ where $O_i^*\in\mathscr{T}$. It follows that $O_2^*\cap A\notin\mathscr{T}$ and hence $O_2^*\not\subset A$. Thus $O_2^*\cap\mathscr{C}A\neq\phi$ and since $\mathscr{C}A$ is indiscrete, it follows that $O_2^*\supset\mathscr{C}A$. Thus $\mathscr{C}A\subset O_2^*$ and $O_2^*\cap A\notin\mathscr{T}$. Hence $X=A^\circ\cup O_2^*$. Now $O_i^*\cap\mathscr{C}A=\phi$. If not, then $O_1^*\supset\mathscr{C}A$ and $U^*=O_1^*\cup(O_2^*\cap A)=O_1^*\cup(O_2^*-\mathscr{C}A)=O_1^*\cup O_2^*\in\mathscr{T}$, a contradiction. Hence $O_1^*\subset A$. Since $X=A^\circ\cup O_2^*$, it follows that $A=(A^\circ\cap A)\cup(O_2^*\cap A)=A^\circ\cup(O_2^*\cap A)=A^\circ\cup O_1^*\cup(O_2^*\cap A)=A^\circ\cup U^*\in\mathscr{U}$.

THEOREM. 6.2. *Let $(X,\mathscr{T})$ be a space and $A\notin\mathscr{T}$. Assume $\mathscr{C}A$ is indiscrete. Then $\mathscr{T}$* imp*$\mathscr{T}(A)$ iff $\mathscr{C}A\subset O\in\mathscr{T}$, $O\cap A\notin\mathscr{T}$ implies that $c(\mathscr{C}A)\subset O$.*

PROOF. Necessity. Suppose $\mathscr{C}A\subset O\in\mathscr{T}$ and $O\cap A\notin\mathscr{T}$. Then by Lemma 6.1, $X=A^\circ\cup O$. But $c(\mathscr{C}A)\subset c(\mathscr{C}A^\circ)=\mathscr{C}A^\circ\subset O$.

Sufficiency. Suppose $\mathscr{C}A\subset O\in\mathscr{T}$ and $O\cap A\notin\mathscr{T}$. By Lemma 1, it suffices to show that $X=A^\circ\cup O$. Now $c(\mathscr{C}A)\subset O$ and $A^\circ\cap c(\mathscr{C}A)=\phi$. Therefore $X=\mathscr{C}A^\circ\cup\mathscr{C}c(\mathscr{C}A)\subset O\cup A^\circ\subset X$.

COROLLARY 6.3. *Let $(X,\mathscr{T})$ be a space and $A\notin\mathscr{T}$. If $\mathscr{C}A$ is indiscrete and $\mathscr{C}A\subset O\in\mathscr{T}$ implies $c(\mathscr{C}A)\subset O$, then $\mathscr{T}$* imp*$\mathscr{T}(A)$.*

COROLLARY 6.4. *Let $(X,\mathscr{T})$ be a space and $x^*\in X$. If $\{x^*\}$ is not closed and $x^*\in O\in\mathscr{T}$ implies that $c(\{x^*\})\subset O$, then $\mathscr{T}$* imp*$\mathscr{T}(\mathscr{C}\{x^*\})$.*

See Theorem 1.2.3 in [1].

COROLLARY 6.5. *Let $(X,\mathscr{T})$ be regular and $\{x^*\}$ not closed. Then $\mathscr{T}$* imp*$\mathscr{T}(\mathscr{C}\{x^*\})$.*

7. Connectedness

THEOREM 7.1. *Let $(X,\mathscr{T})$ be a space, $A\subset X$ and $\mathscr{T}$* imp*$\mathscr{T}(A)$. Suppose*

C_1 and C_2 are separated subsets of $\mathcal{C}A$. Then $c(C_1)\cap c(C_2)\cap A=\phi$.

PROOF. Suppose on the contrary that $x\in c(C_1)\cap c(C_2)\cap A$. Let $O=\mathcal{C}c(C_2)$. We will show that (1) $\mathcal{T}\subset\mathcal{T}(A\cup O)$ and $\mathcal{T}\neq\mathcal{T}(A\cup O)$ and (2) $\mathcal{T}(A\cup O)\subset\mathcal{T}(A)$ and $\mathcal{T}(A\cup O)\neq\mathcal{T}(A)$. (1) and (2) imply that $\mathcal{T}\,\mathrm{imp}\,\mathcal{T}(A)$ is false.

(1) It suffices to show that $A\cup O\notin\mathcal{T}$. Suppose $A\cup O\in\mathcal{T}$. Then $x\in O^*\subset A\cup O$, $O^*\in\mathcal{T}$. But $O^*\cap C_2\neq\phi$: take $y\in O^*\cap C_2$. Then $y\in c(C_2)$. Now $y\notin A$ and hence $y\in O=\mathcal{C}c(C_2)$. Thus $y\in c(C_2)\cap\mathcal{C}c(C_2)$, a contradiction.

(2) Since $A\cup O\in\mathcal{T}(A)$, it follows from Theorem 2.2 that $\mathcal{T}(A\cup O)\subset\mathcal{T}(A)$. It suffices then to show that $A\notin\mathcal{T}(A\cup O)$. Suppose that $A\in\mathcal{T}(A\cup O)$. Then there exist O_1 and O_2 in $\mathcal{T}$ such that $A=O_1\cup(O_2\cap(A\cup O))$. Now $x\notin O_1$ and $x\notin O_2\cap O$ lest $x\notin c(C_1)$. Therefore $x\in O_2\cap A$ and hence $O_2\cap C_1\neq\phi$. Take $z\in O_2\cap C_1$; then $z\in\mathcal{C}c(C_2)=O$. Thus $z\in O_2\cap O\subset A$. Hence $z\in C_1\cap A$. But $C_1\cap A=\phi$, a contradiction.

COROLLARY 7.2. *Let $(X,\mathcal{T})$ be a space and $\mathcal{T}\,\mathrm{imp}\,\mathcal{T}(A)$. If $\mathcal{C}A=C_1\cup C_2$ where C_1 and C_2 are nonempty separated sets, then* (1) *C_1 or C_2 is closed (but not both) and* (2) *$\mathcal{C}A^\circ$ is disconnected.*

PROOF. C_1 and C_2 cannot both be closed lest $A\in\mathcal{T}$. Now $\mathcal{C}A\neq c(\mathcal{C}A)=c(C_1)\cup c(C_2)$. Take $x\in A\cap c(\mathcal{C}A)$. Assume $x\in c(C_1)$. Then $x\notin C_1$ and hence C_1 is not closed. Now $A^\circ\cap c(C_1)=\phi$ and hence $x\in(A-A^\circ)\cap c(C_1)$. Since $A-A^\circ$ is indiscrete by (a) of Theorem 3.2, it follows that $A-A^\circ\subset c(C_1)$. Then $(A-A^\circ)\cap c(C_2)=\phi$ lest $A\cap c(C_1)\cap c(C_2)\supset A-A^\circ\neq\phi$ contradicting Theorem 7.1. Hence $c(C_2)\subset\mathcal{C}A=C_1\cup C_2$. Thus $c(C_2)\subset C_2$ and C_2 is closed. From the proof of (1), it follows that $\mathcal{C}A^\circ=c(C_1)\cup C_2$ and hence $\mathcal{C}A^\circ$ is disconnected.

COROLLARY 7.3. *Let $(X,\mathcal{T})$ be a space and $\mathcal{T}\,\mathrm{imp}\,\mathcal{T}(A)$. Suppose $\mathcal{C}A^\circ$ is disconnected. Then $\mathcal{C}A$ is disconnected.*

PROOF. Let $\mathcal{C}A^\circ=C_1\cup C_2$ where C_1 and C_2 are disjoint nonempty closed sets. Now $A-A^\circ$ is indiscrete and contained in $\mathcal{C}A^\circ$. We may assume $A-A^\circ\subset C_1$. But $A-A^\circ\neq C_1$ lest $\mathcal{C}A=C_2$ and $A\in\mathcal{T}$. Then $\mathcal{C}A=(C_1-(A-A^\circ))\cup C_2$ and $\mathcal{C}A$ is disconnected.

COROLLARY 7.4 *Let $(X,\mathcal{T})$ be a space and $\mathcal{T}\,\mathrm{imp}\,\mathcal{T}(A)$. Then $\mathcal{C}A$ is* *connected iff* $\mathcal{C}A^\circ$ *is connected.*

COROLLARY 7.5. *Let* $(X,\mathcal{T})$ *be a space and* $\mathcal{T}\operatorname{imp}\mathcal{T}(A)$. *If* $A^\circ=\phi$, *then* $\mathcal{C}A$ *is connected iff* X *is connected.*

COROLLARY 7.6. *Let* $(X,\mathcal{T})$ *be a space,* $\mathcal{T}\operatorname{imp}\mathcal{T}(A)$ *and* A *closed. Then* X *is connected iff* A *and* $\mathcal{C}A$ *are connected.*

PROOF. Sufficiency. Suppose $X=O_1\cup O_2$ where O_1 and O_2 are nonempty disjoint open sets. We may assume $A\subset O_1$ and $\mathcal{C}A\subset O_2$. It follows then that $A=O_1\in\mathcal{T}$, a contradiction.

Necessity. We show firstly that $\mathcal{C}A$ is connected. Suppose on the contrary that $\mathcal{C}A$ is not connected. Since $\mathcal{C}A$ is open, then $\mathcal{C}A=O_1\cup O_2$ where O_1 and O_2 are nonempty disjoint open sets. By Corollary 7.2, we may assume that O_1 is closed. Then O_1 is a clopen proper subset of X and X is not connected.

Next we show that A is connected. Suppose on the contrary that $A=E_1\cup E_2$ where E_1 and E_2 are nonempty disjoint closed sets. Since $A-A^\circ$ is indiscrete, we may assume that $A-A^\circ\subset E_1$. Then $E_2=A^\circ\cap\mathcal{C}E_1$ as the reader can verify. Hence E_2 is a proper clopen subset of X.

8. A Closed

THEOREM 8.1. *Let* $(X,\mathcal{T})$ *be a space and* A *closed in* X. *If* $(X,\mathcal{T})$ *is regular, then* $(X,\mathcal{T}(A))$ *is regular.*

This is Theorem 2 in [2]. Note that $\mathcal{T}\operatorname{imp}\mathcal{T}(A)$ is not required here.

EXAMPLE 8.2. Let $X=\{a,b\}$ and $\mathcal{T}=\{\phi,X\}$, $A=\{a\}$. Then $(X,\mathcal{T})$ is regular, $\mathcal{T}\operatorname{imp}\mathcal{T}(A)$, but $(X,\mathcal{T}(A))$ is not regular. Note that A is not closed in $(X,\mathcal{T})$.

THEOREM 8.3. *Let* $(X,\mathcal{T})$ *be a space and* A *a closed subset of* X. *If* $\mathcal{T}$ *has a clopen base, then* $\mathcal{T}(A)$ *has a clopen base.* ($\mathcal{T}\operatorname{imp}\mathcal{T}(A)$ *is not needed here.*)

PROOF. Let $x\in O_1\cup(O_2\cap A)\in\mathcal{T}(A)$.

Case 1. $x\in O_1$. Then there exists a clopen set O^* in $\mathcal{T}$ such that $x\in O^*\subset O_1\subset O_1\cup(O_2\cap A)$. Then O^* is clopen in $\mathcal{T}(A)$.

Case 2. $x\notin O_1$. Then $x\in O_2\cap A$ and hence there exists a clopen set $O^\#$ in $\mathcal{T}$ such that $x\in O^\#\subset O_2$. Thus $x\in O^\#\cap A\subset O_2\cap A\subset O_1\cup(O_2\cap A)$ and $O^\#\cap A$ is clopen in $\mathscr{T}(A)$.

Example 8.2 shows that A closed must be assumed.

THEOREM 8.4. *Let $(X,\mathscr{T})$ be a connected door space. Then $\mathscr{T}$ is maximal relative to connectness.*

PROOF. Let $\mathscr{T}\subset\mathscr{U}$, $\mathscr{T}\neq\mathscr{U}$, $\mathscr{U}$ a topology for X. We will show that $(X,\mathscr{U})$ is not connected. Let $A\in\mathscr{U}-\mathscr{T}$. Then $\mathscr{T}\subset\mathscr{T}(A)\subset\mathscr{U}$. Since $A\notin\mathscr{T}$, then A is closed in $(X,\mathscr{T})$. Thus A is clopen in $\mathscr{T}(A)$ and $(X,\mathscr{T}(A))$ is not connected. It follows then that $(X,\mathscr{U})$ is not connected.

THEOREM 8.5. *Let $(X,\mathscr{T})$ be extremally disconnected, A a closed subset of X and $\mathscr{T}$imp$\mathscr{T}(A)$. Then $\mathscr{T}(A)$ is extremally disconnected.*

PROOF. Let B_1 and $B_2\in\mathscr{T}(A)$, $B_1\cap B_2=\phi$. Let c^* be the closure operator in $\mathscr{T}(A)$. We will show that $c^*(B_1)\cap c^*(B_2)=\phi$.

Case 1. B_1 and B_2 are in $\mathscr{T}$. Then $c^*(B_1)\cap c^*(B_2)\subset c(B_1)\cap c(B_2)=\phi$.

Case 2. $B_1\notin\mathscr{T}$, $B_2\notin\mathscr{T}$. By Theorem 4.4, $B_1=B_1^\circ\cup(A-A^\circ)$ and $B_2=B_2^\circ\cup(A-A^\circ)$. Thus $B_1\cap B_2\supset A-A^\circ\neq\phi$, a contradiction.

Case 3. $B_1\in\mathscr{T}$, $B_2\notin\mathscr{T}$. Then $B_2=B_2^\circ\cup(A-A^\circ)$ again by Theorem 4.4. Now $c^*(B_1)\cap B_2=\phi$ and hence $c^*(B_1)\cap(A-A^\circ)=\phi$. Thus $c^*(B_1)\cap c^*(A-A^\circ)\subset c^*(B_1)\cap c(A-A^\circ)=c^*(B_1)\cap(A-A^\circ)=\phi$. Therefore $c^*(B_1)\cap c^*(B_2)=(c^*(B_1)\cap c^*(B_2^\circ))\cup(c^*(B_1)\cap c^*(A-A^\circ))\subset c(B_1)\cap c(B_2^\circ)=\phi$.

EXAMPLE 8.6. Let $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi, \{b\}, X\}$. Let $A=\{a\}$. Then $(X,\mathscr{T})$ is extremally disconnected, but $\mathscr{T}(A)=\{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ is not. Note that A is not closed nor does $\mathscr{T}$imp$\mathscr{T}(A)$.

9. $A-A^\circ$ Not Closed

THEOREM 9.1. *Let $(X,\mathscr{T})$ be a space, $\mathscr{T}$imp$\mathscr{T}(A)$ and $A-A^\circ$ not closed. If $(X.\mathscr{T})$ is compact (or Lindelof or countably compact), then $(X,\mathscr{T}(A))$ is compact (or Lindelof or countably compact).*

PROOF. We will only prove the compact case. Suppose then that $X=\cup\{B_\alpha:\alpha\in\Delta\}$ where $B_\alpha\in\mathscr{T}(A)$ for all $\alpha\in\Delta$.

Case 1. $B_\alpha\in\mathscr{T}$ for all $\alpha\in\Delta$. Then clearly $X=B_{\alpha_1}\cup\cdots\cup B_{\alpha_k}$ for some $\alpha_i\in\Delta$.

Case 2. $B_{\alpha^*}\notin\mathscr{T}$ for some $\alpha^*\in\Delta$. It follows from Theorem 4.4 that $X=(A-$

$A^\circ)\cup[\cup\{\mathring{B}_\alpha:\alpha\in\Delta\}]$. Since $A-A^\circ$ is not closed, then $(A-A^\circ)\cap\mathring{B}_{\alpha\#}$ for some $\alpha^\#$ in Δ and hence $A-A^\circ\subset\mathring{B}_{\alpha\#}$ since $A-A^\circ$ is discrete. It follows then that $X=\cup\{\mathring{B}_\alpha:\alpha\in\Delta\}$ and each $\mathring{B}_\alpha\in\mathscr{T}$. Compactness of $(X.\mathscr{T}(A))$ is now immediate.

THEOREM 9.2. *Let $(X, \mathscr{T})$ be a space, $\mathscr{T}$ imp $\mathscr{T}(A)$ and $A-A^\circ$ not closed. If $(X, \mathscr{T})$ is sequentially compact, then $(X, \mathscr{T}(A))$ is sequentially compact.*

PROOF. It suffices to show that if $\{x_n\}$ is convergent in the space $(X, \mathscr{T})$, then $\{x_n\}$ is convergent in $(X, \mathscr{T}(A))$. To this end, suppose $\lim x_n=x$ in $(X, \mathscr{T})$, but $\lim x_n=y$ in $(X, \mathscr{T}(A))$ for no $y\in X$. Then for each $y\in X$, there exists a $B_y\in\mathscr{T}(A)$ such that $y\in B_y$ and x_n is not eventually in B_y. But $x\in B_x\in\mathscr{T}(A)-\mathscr{T}$ and hence by Theorem 4.4 $x\in\mathring{B}_x\cup(A-A^\circ)$. Therefore $x\in A-A^\circ$. It follows then that $X=(A-A^\circ)\cup\cup\{\mathring{B}_y:y\in X\}$, and $X=\cup\{B^\circ_y: y\in X\}$ (see the reasoning in Case 2 of Theorem 9.1). But $x\in\mathring{B}_{y*}$ for some y^* and hence x_n is eventually in B_{y*}, a contradiction.

THEOREM 9.3. *Let $(X,\mathscr{T})$ be connected, $\mathscr{T}$ imp $\mathscr{T}(A)$ and $A-A^\circ$ not closed. Then $(X, \mathscr{T}))$ is connected.*

PROOF. Suppose on the contrary that $X=B_1\cup B_2$ where B_1 and B_2 are in $\mathscr{T}(A)$, disjoint and nonempty.

Case 1. B_1 and B_2 are in $\mathscr{T}$. Then X is not connected, a contradiction.

Case 2. $B_1\notin\mathscr{T}$, $B_2\notin\mathscr{T}$. Then by Theorem 4.4, $B_1=\mathring{B}_1\cup(A-A^\circ)$ and $B_2=\mathring{B}_2\cup(A-A^\circ)$ and $B_1\cap B_2\supset A-A^\circ\neq\phi$, a contradiction.

Case 3. $B_1\notin\mathscr{T}$, $B_2\in\mathscr{T}$. Then $B_1=\mathring{B}_1\cup(A-A^\circ)$ and $X=B_2\cup\mathring{B}_1\cup(A-A^\circ)$. Then $A-A^\circ$ is closed, a contradiction.

See Theorem 9 in [2].

THEOREM 9.4. *Let $(X, \mathscr{T})$ be normal, $\mathscr{T}$ imp $\mathscr{T}(A)$ and $A-A^\circ$ not closed. Then $(X,\mathscr{T}(A))$ is normal.*

PROOF. Let $X=B_1\cup B_2$ where B_1 and B_2 are in $\mathscr{T}(A)$.

Case 1. B_1 and B_2 are in $\mathscr{T}$. Then there exist F_1 and F_2 $\mathscr{T}$-closed and hence $\mathscr{T}(A)$-closed such that $X=F_1\cup F_2$, $F_i\subset B_i$.

Case 2. B_1 and B_2 are not in $\mathscr{T}$. By Theorem 4.4, $B_i-\mathring{B}_i=A-A^\circ$ and hence $X=\mathring{B}_1\cup\mathring{B}_2\cup(A-A^\circ)$. It follows then that $(A-A^\circ)$ is closed, a contradiction.

Case 3. $B_1\in\mathscr{T}$, $B_2\in\mathscr{T}(A)-\mathscr{T}$. Then $B_2=\mathring{B}_2\cup(A-A^\circ)$ and $X=\mathring{B}_1\cup\mathring{B}_2\cup$

$(A-A^\circ)$. If $\mathring{B}_1\cap(A-A^\circ)=\phi$, then $A-A^\circ$ is closed, a contradiction. If $\mathring{B}_1\cap$ $(A-A^\circ)\neq\phi$, then $\mathring{B}_1\supset A-A^\circ$ and $X=\mathring{B}_1\cup\mathring{B}_2$. Procede as in Case 1.

See Theorem 5 in [2].

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