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PAIRWISE C-COMPACT SPACES

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1. Introduction

In a recent paper Cooke and Reilly [2] discussed various notions of bitopological compactness and the relationships between them. In this paper we introduce another independent notion of compactness which generalizes Viglino's C-compactness [10] to bitopological spaces. It turns out to be more general than Fletcher, Hoyle and Patty's pairwise compactness [3] and in common with the latter is not product invariant. It is invariant under pairwise continuous surjections, has the "maximal compact and minimal Hausdorff" property and may be characterized in terms of the adherent convergence of certain open filter bases. We follow the terminology of Cooke and Reilly [2] and Kelly [6].

2. Pairwise C-compactness

For topological spaces it is well known that every continuous function from a compact space to a Hausdorff space is closed. It was shown by Viglino [10] that this property holds for a larger class of spaces, called the C-compact spaces and these spaces have been a fruitful source of research (see eg. [4], [5], [7], [10], [11], [12]). The bitopological analogue of the above mentioned classical result is the following:

PROPOSITION 1. *Let* $(X,\mathscr{P},\mathscr{Q})$ *be pairwise compact,* $(X^*,\mathscr{P}^*,\mathscr{Q}^*)$ *be pairwise Hausdorff and let* f: $(X,\mathscr{P},\mathscr{Q})\rightarrow(X^*,\mathscr{P}^*,\mathscr{Q}^*)$ *be pairwise continuous. Then if* $A\neq X$ *is a* $\mathscr{P}$*-closed (resp.* $\mathscr{Q}$*-closed) subset of* X, $f(A)$ *is a* $\mathscr{P}^*$*-closed (resp.* $\mathscr{Q}^*$*-closed) subset of* X^*.

PROOF. Since $A\neq X$ is $\mathscr{P}$-closed it is $\mathscr{Q}$-compact (see [3], lemma 3). Hence $f(A)$ is $\mathscr{Q}^*$-compact and it follows that $f(A)$ is $\mathscr{P}^*$-closed.

The class of *pairwise C-compact spaces*, defined below, properly contains the class of pairwise compact bitopological spaces and have the property in proposition 1 above.

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DEFINITION 1. In a bitopological space $(X, \mathscr{P}, \mathscr{Q})$ we say that *$\mathscr{P}$ is C-compact with respect to $\mathscr{Q}$* if for each $\mathscr{P}$-closed set $A \neq X$ and any $\mathscr{Q}$-open cover $\mathscr{V}$ of A there exists a finite number of members of V, say $V_1, V_2, \cdots, V_n$ such that

$$A \subset \mathrm{cl}_{\mathscr{P}}(\bigcup_{i=1}^{n} V_i).$$

If in addition $\mathscr{Q}$ is C-compact with respect to $\mathscr{P}$ we say $(X, \mathscr{P}, \mathscr{Q})$ is *pairwise C-compact.*

PROPOSITION 2. *If $(X, \mathscr{P}, \mathscr{Q})$ is pairwise compact it is pairwise C-compact.*

The following is an example of a pairwise C-compact bitopological space which is not pairwise compact.

EXAMPLE 1. Let $X = [0, 1]$,

$\mathscr{P} = \{\phi, X\} \cup \{[0, b) \mid b \in [0, 1]\}$,

$\mathscr{Q} = \{\phi, X, \{1\}\}$.

Let $B \neq X$ be any $\mathscr{Q}$-closed subset of X-the only non-trivial such set is $B = [0, 1)$, Let $\mathscr{U}$ be any $\mathscr{P}$-open cover of B. If $\mathscr{U}$ does not contain X or $[0, 1)$ choose any $[0, b)$, $0 < b < 1$, in $\mathscr{U}$ and observe that $\mathrm{cl}_{\mathscr{Q}}[0, b) = [0, 1) = B$. Thus $\mathscr{Q}$ is C-compact with respect to $\mathscr{P}$ and clearly then $(X, \mathscr{P}, \mathscr{Q})$ is pairwise C-compact. However, the pairwise open cover $\{[0, b) \mid b \in [0, 1]\} \cup \{1\}$ of X has no finite subcover so $(X, \mathscr{P}, \mathscr{Q})$ is not pairwise compact.

PROPOSITION 3. *Let $f: (X, \mathscr{P}, \mathscr{Q}) \to (X^*, \mathscr{P}^*, \mathscr{Q}^*)$ be a pairwise continuous map from a pairwise C-compact space to a pairwise Hausdorff space. Then for any $\mathscr{P}$-closed (resp. $\mathscr{Q}$-closed) subset $A \neq X$, $f(A)$ is $\mathscr{P}^*$-closed (resp. $\mathscr{Q}^*$-closed).*

PROOF. Let $A \neq X$ be $\mathscr{P}$-closed and suppose $a \notin f(A)$. For each $y \in f(A)$ choose a $\mathscr{Q}^*$-open neighbourhood V_y such that $a \notin \mathrm{cl}_{\mathscr{P}^*} V_y$. The family $\{f^{-1}(V_y) \mid y \in f(A)\}$ is a $\mathscr{Q}$-open cover of A and since $\mathscr{P}$ is C-compact with respect to $\mathscr{Q}$ there exists a positive integer n such that

$$A \subset \mathrm{cl}_{\mathscr{P}}(\bigcup_{i=1}^{n} f^{-1}(V_{y_i}))$$

By continuity of f we then have

$$f(A) \subset \mathrm{cl}_{\mathscr{P}^*}(\bigcup_{i=1}^{n} V_{y_i}).$$

Thus $X^* - \mathrm{cl}_{\mathscr{P}^*}(\bigcup_{i=1}^{n} V_{y_i})$ is a $\mathscr{P}^*$-open neighbourhood of a, disjoint from $f(A)$ and hence $f(A)$ is $\mathscr{P}^*$-closed.

COROLLARY 1. *If $(X, \mathscr{P}, \mathscr{Q})$ is pairwise C-compact, $(X^*, \mathscr{P}^*, \mathscr{Q}^*)$ pairwise Hausdorff and if f: $(X, \mathscr{P}, \mathscr{Q}) \to (X^*, \mathscr{P}^*, \mathscr{Q}^*)$ is a pairwise continuous bijection, then f is a pairwise homeomorphism.*

COROLLARY 2. *If $(X, \mathscr{P}, \mathscr{Q})$ is pairwise Hausdorff and pairwise C-compact, then it is minimal pairwise Hausdorff and maximal pairwise C-compact.*

PROPOSITION 4. *The pairwise continuous image of a pairwise C-compact space is pairwise C-compact.*

PROOF. Let f: $(X, \mathscr{P}, \mathscr{Q},) \to (X^*, \mathscr{P}^*, \mathscr{Q}^*)$ be a pairwise continuous surjection and let $(X, \mathscr{P}, \mathscr{Q})$ he pairwise C-compact. Let $A \neq X^*$ be $\mathscr{P}^*$-closed and let $\mathscr{V}$ be any $\mathscr{Q}^*$-open cover of A. Then $\{f^{-1}(V) \mid V \in \mathscr{V}\}$ is a $\mathscr{Q}$-open cover of the $\mathscr{P}$-closed set $f^{-1}(A) \neq X$ so there exist a finite number $V_1, V_2, \cdots, V_n$ of members of $\mathscr{V}$ with

$$f^{-1}(A) \subset \mathrm{cl}_{\mathscr{P}}(\bigcup_{i=1}^{n} f^{-1}(V_i)) \subset f^{-1}(\mathrm{cl}_{\mathscr{P}^*}(\bigcup_{i=1}^{n} V_i))$$

which gives $A \subset \mathrm{cl}_{\mathscr{P}^*}(\bigcup_{i=1}^{n} V_i)$.

COROLLARY 3. *If the product of a non-empty family of non-empty bitopological spaces $(X_\alpha, \mathscr{P}_\alpha, \mathscr{Q}_\alpha)$ is pairwise C-compact then each $(X_\alpha, \mathscr{P}_\alpha, \mathscr{Q}_\alpha)$ is pairwise C-compact.*

The converse is false as the following example shows.

EXAMPLE 2. Let X_i denote the real line, $\mathscr{P}_i$ and $\mathscr{Q}_i$ $(i=1, 2)$ the topologies on X_i having as open bases all sets of the form $(-\infty, a)$ and (a, ∞) respectively. Let $(X_1 \times X_2, \mathscr{P}, \mathscr{Q})$ be the associated product bitopological space ([9], 3.8). Then $(X_i, \mathscr{P}_i, \mathscr{Q}_i)$ is pairwise C-compact (in fact pairwise compact) but $(X_1 \times X_2, \mathscr{P}, \mathscr{Q})$ is not pairwise C-compact: Let $A = (X_1 \times X_2) - [(-\infty, k) \times (-\infty, k)]$, k any real number. For each positive integer n, let $V_n = (-n, \infty) \times (-n, \infty)$. The $\mathscr{Q}$-open cover $\{V_n\}$ of the $\mathscr{P}$-closed set A has no finite subfamily such that the $\mathscr{P}$-closure of the union of its members contains A. Hence pairwise C-compactness is not product invariant.

Viglino [11] characterized C-compactness is terms of adherent convergence of open filter bases. The bitopological analogue of this result is discussed below.

DEFINITION 2. A filter base $\mathscr{F}$ on $(X, \mathscr{P}, \mathscr{Q})$ is said to be *$\mathscr{P}$-adherent convergent with respect to $\mathscr{Q}$* if every non-empty $\mathscr{P}$-open neighbourhood of the $\mathscr{Q}$-adherent set of $\mathscr{F}$ contains a member of $\mathscr{F}$.

PROPOSITION 5. *In a bitopological space $(X, \mathscr{P}, \mathscr{Q})$, $\mathscr{P}$ is C-compact with respect to $\mathscr{Q}$ if and only if every $\mathscr{P}$-open filter base $\mathscr{F}$ on X is $\mathscr{P}$-adherent convergent with respect to $\mathscr{Q}$.*

PROOF. Let $\mathscr{F}$ be a $\mathscr{P}$-open filter base on X and suppose $\mathscr{P}$ is C-compact with respect to $\mathscr{Q}$. Let G be any non-empty $\mathscr{P}$-open set with

$$\bigcap \{\mathrm{cl}_{\mathscr{Q}} F \mid F \in \mathscr{F}\} \subset G.$$

Since $\{X - \mathrm{cl}_{\mathscr{Q}} F \mid F \in \mathscr{F}\}$ is a $\mathscr{Q}$-open cover of the $\mathscr{P}$-closed set $X - G \neq X$ there exist $F_1, F_2, \cdots, F_n$ in F with

$$\bigcup_{i=1}^{n} \{\mathrm{cl}_{\mathscr{P}}(X - \mathrm{cl}_{\mathscr{Q}} F_i)\} \supset X - G$$

Hence

$$\bigcap_{i=1}^{n} \{X - \mathrm{cl}_{\mathscr{P}}(X - \mathrm{cl}_{\mathscr{Q}} F_i)\} \subset G$$

and since the F_i's are $\mathscr{P}$-open it follows that

$$\bigcap_{i=1}^{n} F_i \subset G.$$

Conversely, if $\mathscr{P}$ is not C-compact with respect to $\mathscr{Q}$ then there exists a $\mathscr{Q}$-open cover $\mathscr{V}$ for a $\mathscr{P}$-closed set $A \neq X$ such that A is not contained in the $\mathscr{P}$-closure of a finite union of members of $\mathscr{V}$. Letting $\mathscr{F} = \{X - \mathrm{cl}_{\mathscr{P}} V \mid V \in \mathscr{V}\}$ we obtain a $\mathscr{P}$-open filter base on X (we may assume that V is closed under finite unions). Furthermore, no member of $\mathscr{F}$ is contained in $X - A$ and the $\mathscr{Q}$-adherent set of $\mathscr{F}$ is contained in $X - A$.

COROLLARY 4. *$(X, \mathscr{P}, \mathscr{Q})$ is pairwise C-compact if and only if every $\mathscr{P}$-open filter base on X is $\mathscr{P}$-adherent convergent with respect to $\mathscr{Q}$ and every $\mathscr{Q}$-open filter base on X is $\mathscr{Q}$-adherent convergent with respect to $\mathscr{P}$.*

The following is an example of a B-compact (Birsan [1], definition 1) and hence by theorem 3 in [2] a *pseudo-compact* (Seagrove [8], definition 4.1) bitopological space which is not pairwise C-compact.

EXAMPLE 3. Let $X=[0,1]$,

$$\mathscr{P}=\{\phi,\ X,\ \{0\}\}\cup\{[0,\ a)\,|\,a\in[0,\ 1]\},$$
$$\mathscr{Q}=\{\phi,\ X,\ \{1\}\}\cup\{(a,\ 1]\,|\,a\in[0,\ 1]\}.$$

$(X,\ \mathscr{P},\ \mathscr{Q})$ is B-compact ([2], example 3) but $\mathscr{P}$ is not C-compact with respect to $\mathscr{Q}$: Consider the $\mathscr{P}$-closed set $A=(0,1]$ and the $\mathscr{Q}$-open cover $\mathscr{V}=\{(a,\ 1]\,|\,a\in[0,\ 1]\}$ of A. Any union over a finite subfamily of $\mathscr{V}$ is of the form $(a,\ 1]$, some $0<a<1$ and clearly $\mathrm{cl}_{\mathscr{P}}(a,\ 1]\not\supset A$.

In view of the results in [2] and in this paper the following implications hold in general. Furthermore no arrows can be reversed and no others fitted:

finite→*semi-compact*→*pairwise compact*→*pairwise C-compact*

semi-compact ↓ *pseudo-compact*

B-compact ↑ *pseudo-compact*

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