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ANTI-HOLOMORPHIC SUBMANIFOLDS OF A SASAKIAN MANIFOLD WITH VANISHING C-BOCHNER CURVATURE TENSOR.

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In [9], Yano proved

THEOREM A. *Let M^n, $n \geqq 5$, be an anti-invariant submanifold of a Sasakian manifold M^{2n-1} with vanishing C-Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is locally a product of a conformally flat Riemannian space and a 1-dimensional space.*

THEOREM B. *Let M^n, $n \geqq 4$, be a totally umbilical anti-invariant submanifold normal to the structure vector field ξ of a Sasakian manifold M^{2n+1} with vanishing C-Bochner curvature tensor. Then M^n is conformally flat.*

THEOREM C. *Let M^n, $n \geqq 4$, be an anti-invariant submanifold normal to the structure vector field ξ of a Sasakian manifold M^{2n+1} with vanishing C-Bochner curvature tensor. If the second fundamental tensors commute, then M^n is conformally flat.*

The purpose of the present paper is to prove the following Theorem 1, 2 and 3 corresponding to Theorems A, B and C by replacing the condition that the submanifold is anti-invariant with that is anti-holomorphic respectively.

THEOREM 1. *Let M^n, $n \geqq 5$, be an anti-holomorphic submanifold tangent to the structure vector field ξ of a Sasakian manifold M^{2n-1} with vanishing C-Bochner curvature tensor. If the second fundamental tensors of M^n commute, then M^n is locally a product of a conformally flat Riemannian space and a 1-dimensional space.*

THEOREM 2. *Let M^n, $n \geqq 4$, be a totally umbilical anti-holomorphic submanifold of a Sasakian manifold M^{2n+1} with vanishing C-Bochner curvature tensor. Then M^n is conformally flat.*

THEOREM 3. *Let M^n, $n \geqq 4$, be an anti-holomorphic submanifold of a Sasakian manifold M^{2n+1} with vanishing C-Bochner curvature tensor. If the second fund-*

amental tensors of M^n commute, then M^n is conformally flat.

1. C-Bochner curvature tensor

We first of all recall definition and fundamental properties of Sasakian manifolds for later use. Let M^{2m+1} be a $(2m+1)$-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods $\{U\,;\,y^\kappa\}$ (the indices $\alpha, \beta, \cdots, \kappa, \lambda, \mu, \cdots$ run over the range $\{1, 2, \cdots, 2m+1\}$) in which there are given a tensor field ϕ_λ^κ of type (1,1), a vector field ξ^κ, a 1-form η_λ and a Riemannian metric tensor $g_{\mu\lambda}$ satisfying

$$(1.1)\qquad \phi_\lambda^\kappa\,\phi_\mu^\lambda=-\delta_\mu^\kappa+\eta_\mu\xi^\kappa,\ \ \phi_\lambda^\kappa\,\xi^\lambda=0,\ \ \eta_\lambda\phi_\mu^\lambda=0,\ \ \eta_\lambda\xi^\lambda=1,$$
$$g_{\gamma\beta}\,\phi_\mu^\gamma\,\phi_\lambda^\beta=g_{\mu\lambda}-\eta_\mu\eta_\lambda,\ \ \eta_\lambda=g_{\lambda\kappa}\,\xi^\kappa.$$

If

$$(1.2)\qquad \nabla_\lambda\,\xi^\kappa=\phi_\lambda^\kappa,\ \ \nabla_\mu\,\phi_\lambda^\kappa=-g_{\mu\lambda}\xi^\kappa+\delta_\mu^\kappa\,\xi_\lambda,$$

where ∇_λ denotes the operator of covariant differentiation with respect to $g_{\mu\lambda}$, then such a set $(\phi_\lambda^\kappa,\ \xi^\kappa,\ \eta_\lambda,\ g_{\mu\lambda})$ is called a *normal contact structure.* Such a manifold M^{2m+1} is called a *Sasakian manifold.* In view of the last equation of (1.1) we shall write ξ_λ instead of η_λ in the sequel. In a Sasakian manifold, the tensor field $\phi_{\mu\lambda}=\phi_\mu^\alpha\,g_{\alpha\lambda}$ is skew-symmetric.

It is well known that in a Sasakian manifold equation (1.2) and the Ricci identity give

$$(1.3)\qquad K_{\mu\nu\lambda}{}^\kappa\xi^\lambda=\delta_\mu^\kappa\,\xi_\nu-\delta_\nu^\kappa\xi_\mu,$$

$$(1.4)\qquad K_{\mu\lambda}\,\xi^\lambda=2m\,\xi_\mu,$$

$$(1.5)\qquad K_{\mu\alpha}\,\phi_\lambda^\alpha+K_{\kappa\alpha}\,\phi_\mu^\alpha=0,$$

where $K_{\mu\nu\lambda}{}^\kappa$ and $K_{\mu\lambda}$ are the curvature tensor and the Ricci tensor of the manifold respectively.

As an analogue of the Bochner curvature tensor in a Kaehlerian manifold, the C-Bochner curvature tensor in a Sasakian manifold is defined (cf. [9]) by

$$(1.6)\qquad B_{\nu\mu\lambda}{}^\kappa=K_{\nu\mu\lambda}{}^\kappa+(\delta_\nu^\kappa-\xi_\nu\xi^\kappa)L_{\mu\lambda}-(\delta_\mu^\kappa-\xi_\mu\xi^\kappa)L_{\nu\lambda}$$
$$+L_\nu^\kappa(g_{\mu\lambda}-\xi_\mu\,\xi_\lambda)-L_\mu^\kappa(g_{\nu\lambda}-\xi_\nu\,\xi_\lambda)$$

$$+\phi_\nu^\kappa M_{\mu\lambda}-\phi_\mu^\kappa M_{\nu\lambda}+M_\nu^\kappa \phi_{\nu\lambda}-M_\mu^\kappa \phi_{\nu\lambda}-2(\phi_{\nu\mu}M_\lambda^\kappa+M_{\nu\mu}\phi_\lambda^\kappa)$$
$$+(\phi_\nu^\kappa \phi_{\mu\lambda}-\phi_\mu^\kappa \phi_{\nu\lambda}-2\phi_{\nu\mu}\phi_\lambda^\kappa),$$

where

(1.7) $$L_{\mu\lambda}=\frac{1}{2(m+2)}[-K_{\mu\lambda}-(L+3)g_{\mu\lambda}+(L-1)\xi_\mu \xi_\lambda],\quad L_\mu^\kappa=L_{\mu\alpha}g^{\alpha\kappa},$$

(1.8) $$L=g^{\mu\lambda}L_{\mu\lambda},$$

(1.9) $$M_{\mu\lambda}=-L_{\mu\alpha}\phi_\lambda^\alpha,\quad M_\nu^\kappa=M_{\nu\alpha}g^{\alpha\kappa}.$$

From (1.7) and (1.8), we have

(1.10) $$L=-\frac{K+2(3m+2)}{4(m+1)},$$

where K is the scalar curvature of the manifold.

Using (1.4), we have from (1.7)

(1.11) $$L_{\mu\lambda}\xi^\lambda=-\xi_\mu.$$

From the first equation of (1.9) and (1.11), we have

(1.12) $$M_{\mu\alpha}\phi_\lambda^\alpha=L_{\mu\lambda}+\xi_\mu\xi_\lambda.$$

It is easily verify that the C-Bochner curvature tensor satisfies the following identities:

$$B^\kappa_{\nu\mu\lambda}=-B_{\nu\lambda\kappa}{}^\mu,\quad B_{\mu\lambda\kappa}{}^\nu+B_{\mu\lambda\nu}{}^\kappa+B_{\lambda\nu\mu}{}^\kappa=0,\quad B_{\alpha\mu\lambda}{}^\alpha=0,\quad B_{\nu\mu\lambda\kappa}=-B_{\nu\mu\kappa\lambda},$$
$$B_{\nu\mu\lambda\kappa}=B_{\lambda\kappa\nu\mu},\quad B_{\nu\mu\lambda}{}^\kappa\xi_\kappa=0,\quad B_{\nu\mu\alpha}{}^\kappa\phi_\lambda^\alpha=B_{\nu\mu\lambda}{}^\alpha\phi_\alpha^\kappa,\quad B_{\nu\mu\lambda}{}^\kappa\phi^{\nu\mu}=0,$$

where $B_{\nu\mu\lambda\kappa}=B_{\nu\mu\lambda}{}^\alpha g_{\alpha\kappa}$

2. Anti-holomorphic submanifolds of a Sasakian manifold

We consider an n-dimensional Riemannian manifold M^n, $n>1$, covered by a system of coordinate neighborhoods $\{V; y^h\}$ (the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, n\}$) and isometrically immersed in a Sasakian manifold M^{2m+1} and denote the immersion by

(2.1) $$x^\kappa=x^\kappa(y^h).$$

We put

(2.2) $$B_i^\kappa=\partial_i x^\kappa \quad (\partial_i=\partial/\partial y^i)$$

and denote by C_y^κ $2m+1-n$ mutually orthogonal unit vectors normal to M^n (the indices x, y, z run over the range $\{(n+1), \cdots, (2m+1)\}$). Then the metric tensor g_{ji} of M^n and that of the normal bundle are respectively given by

$$g_{ji}=g_{\mu\lambda}B_{ji}^{\mu\lambda},\ g_{zy}=g_{\lambda\mu}C_{zy}^{\mu\lambda},$$

where $B_{ji}^{\mu\lambda}=B_j^\mu\ B_i^\lambda$ and $C_{zy}^{\mu\lambda}=C_z^\mu\ C_y^\lambda$.

If the transform by ϕ_λ^κ of any normal vector to M^n is orthogonal to itself, the submanifold M^n is called *anti-holomorphic* in M^{2m+1}. Since the rank of ϕ_λ^κ is $2m$, we have $2m+1-n-1\leqq n$, that is, $m\leqq n$.

For an anti-holomorphic submanifold M^n in M^{2m+1}, we have equations of the form

(2.3) $$\phi_\lambda^\kappa B_i^\lambda=f_i^h B_h^\kappa-f_i^x C_x^\kappa,$$

(2.4) $$\phi_\lambda^\kappa C_y^\lambda=f_y^i B_i^\kappa,$$

(2.5) $$\xi^\kappa=\xi^i B_i^\kappa+\xi^x C_x^\kappa.$$

Using $\phi_{\mu\lambda}=-\phi_{\lambda\mu}$, we have, from (2.3) and (2.4),

(2.6) $$f_{ix}=f_{xi},\ f_{ij}=-f_{ji},$$

where $f_{ix}=f_i^z g_{zx}$, $f_{xi}=f_x^j g_{ji}$ and $f_{ij}=f_i^h g_{hj}$.

Applying ϕ to (2.3), (2.4) and (2.5) and using (1.1) and these equations we find

(2.7)
$$\begin{cases} \text{(i)}\ \ f_i^x f_x^j=\delta_i^j-\xi_i\xi^j+f_i^h f_h^j, & \\ \text{(ii)}\ \ f_y^i f_i^x=\delta_y^x-\xi_y\xi^x, & \text{(iii)}\ \ f_i^h f_h^x=-\xi_i\xi^x, \\ \text{(iv)}\ \ f_y^i f_i^h=\xi_y\xi^h, & \text{(v)}\ \ f_i^h\xi^i=-f_x^h\xi^x, \\ \text{(vi)}\ \ f_i^x\xi^i=0, & \text{(vii)}\ \ \xi_i\xi^i+\xi_x\xi^x=1, \end{cases}$$

where $\xi_i=g_{ih}\xi^h$ and $\xi_x=g_{yx}\xi^y$, (vii) be a direct consequence of $\xi_\lambda\xi^\lambda=1$.

Differentiating (2.3), (2.4) and (2.5) covariantly along M^n and using (1.2), (2.7), equations of Gauss and those of Weingarten

$$\nabla_j B_i^\kappa=h_{ji}{}^x C_x^\kappa,\ \nabla_j C_x^\kappa=-h_j{}^i{}_x B_i^\kappa,$$

where ∇_j denotes the operator of covariant diffentiation along M^n and $h_{ji}{}^x$ and $h_j{}^i{}_x=h_{jt}{}^z g^{ti} g_{zx}$ are the second fundamental tensors of M^n with respect to normals

C^{κ}_{x}, we find

$$(2.8)\quad \begin{cases} \text{(i) } \nabla_j f_i^h = \delta_i^h \xi_i - g_{ji}\xi^h + h_{ji}{}^x f_x^h - f_i^x h_j{}^h{}_x, & \text{(ii) } \nabla_j f_i^x = g_{ji}\xi^x + h_{jh}{}^x f_h^i, \\ \text{(iii) } \nabla_j f_y{}^i = \delta_j{}^i \xi_y - h_j{}^h{}_y f_h^i, & \text{(iv) } h_{ji}{}^x f_y^i = h_j{}^i{}_y f_i^x, \\ \text{(v) } \nabla_j \xi^i = h_j{}^i{}_x \xi^x + f_j^i, & \text{(vi) } \nabla_j \xi^x = -f_j^x - h_{ji}{}^x \xi^i. \end{cases}$$

I. *The case in which ξ^{κ} is tangent to M^n.*

Now suppose that ξ^{κ} is tangent to M^n, that is, $\xi^x=0$. From (2.7), (i) and (ii) we find

$$-f_{ij}f^{ji}=2(m+1-n).$$

Thus, if $n=m+1$, we have $f_{ji}=0$, and (2.7) and (2.8) respectively reduces to

$$(2.9)\quad \begin{cases} \text{(i) } f_i^x f_x^j = \delta_i^j - \xi_i \xi^j, & \text{(ii) } f_y^i f_i^x = \delta_y^x, \\ \text{(iii) } f_i^x \xi^i = 0, & \text{(iv) } \xi_i \xi^i = 1 \end{cases}$$

and

$$(2.10)\quad \begin{cases} \text{(i) } \delta_j^h \xi_i - g_{ji}\xi^h + h_{ji}{}^x f_x^h - f_i^x h_j{}^h{}_x = 0, & \text{(ii) } \nabla_j f_i^x = 0, \\ \text{(iii) } h_{ji}{}^x f_y^i = h_j{}^i{}_y f_i^x, \quad \text{(iv) } \nabla_j \xi^i = 0, & \text{(v) } f_j^x + h_{ji}{}^x \xi^i = 0. \end{cases}$$

Equation (2.10), (i) shows that an anti-holomorphic submanifold tangent to ξ^{κ} cannot be totally umbilical or totally contact umbilical. Because if $h_{ji}{}^x$ is of the from $(\alpha g_{ji}+\beta\xi_j\xi_i)h^x$, then from (2.10), (i) we have

$$(n-1)\xi_i=(n-1)\alpha h_x f_i^x+\beta h_x f_i^x,$$

and consequently, transvecting with ξ^i and using (2.9), (iii) gives $(n-1)\xi_i\xi^i=0$, which is a contradiction for $n>1$.

From (2.10), (ii) and the Ricci identity we find

$$(2.11)\qquad K_{kji}{}^h f_h^x = K_{kjy}{}^x f_i^y,$$

where $K_{kji}{}^h$ is the curvature tensor of M^n and $K_{kjy}{}^x$ that of the normal bundle of M^n.

Taking account of (2.9), (i), (ii) and (2.11) yields

$$(2.12)\qquad K_{kji}{}^h = K_{kjy}{}^x f_y^i f_h^x, \quad K_{kjy}{}^x = K_{kji}{}^h f_y^i f_h^x$$

with the help of $K_{kji}{}^{h}\xi^{i}=0$. Equations (2.12) shows that $K_{kji}{}^{h}=0$ and $K_{kjy}{}^{x}=0$ are equivalent to each other.

II. *The case in which ξ^{κ} is normal to M^{n}.*

From (2.7), (i), (ii) and using (2.7), (vii) we have

$$2\xi_{i}\xi^{i}+f_{ij}f^{ij}=-2(m-n).$$

Suppose $n=m$ we have $f_{ij}=0$ and $\xi^{i}=0$, that is, ξ^{κ} is normal to M^{n}. Then (2.7) and (2.8) respectively become

$$(2.13)\quad \begin{cases} \text{(i) } f_{i}^{x}f_{x}^{j}=\delta_{j}^{i}, & \text{(ii) } f_{y}^{i}f_{i}^{x}=\delta_{y}^{x}-\xi_{y}\xi^{x}, \\ \text{(iii) } f_{x}^{h}\xi^{x}=0, & \text{(iv) } \xi_{x}\xi^{x}=1 \end{cases}$$

and

$$(2.14)\quad \begin{cases} \text{(i) } h_{ji}{}^{x}f_{x}^{h}=h_{j}{}^{h}{}_{x}f_{i}^{x}, & \text{(ii) } \nabla_{j}f_{i}^{x}=g_{ji}\xi^{x}, \\ \text{(iii) } \nabla_{j}f_{y}^{i}=\delta_{j}^{i}\xi_{y}, & \text{(iv) } h_{ji}{}^{x}f_{y}^{i}=h_{j}{}^{i}{}_{y}f_{i}^{x}, \\ \text{(v) } h_{j}{}^{i}{}_{x}\xi^{x}=0, & \text{(vi) } \nabla_{j}\xi^{x}=-f_{i}^{x}. \end{cases}$$

Suppose that M^{n} is totally umbilical and put $h_{ji}{}^{x}=g_{ji}h^{x}$. Then from (2.14), (i) we have

$$g_{ji}h^{x}f_{x}^{h}=\delta_{j}^{h}h_{x}f_{i}^{x},$$

from which $f_{x}^{h}h^{x}=0$, for $n>1$. From (2.14), (iv) we have

$$h^{x}f_{yj}=h_{y}f_{j}^{x},$$

from which transvecting with h^{y} and using $f_{yj}h^{y}=0$ give $h_{y}h^{y}f_{j}^{x}=0$ and consequently $h_{y}h^{y}=0$, that is, $h_{y}=0$. Thus M^{n} must be totally geodesic.

From (2.14), (ii) and (vi), we find

$$\nabla_{j}\nabla_{i}\xi^{x}=-g_{ji}\xi^{x},$$

from which, using the Ricci identity,

$$K_{kjy}{}^{x}\xi^{y}=0.$$

On the other hand, from (2.14), (ii) and (vi), we have

$$-K_{kji}{}^{h}f_{h}^{x}+K_{kjy}{}^{x}f_{i}^{y}=-f_{k}^{x}g_{ji}+f_{j}^{x}g_{ki},$$

from which, using (2.13), (i),

(2.15) $$K_{kji}{}^{h}=K_{kjy}{}^{x}f_{i}^{y}f_{x}^{h}+\delta_{k}^{h}g_{ji}-\delta_{j}^{h}g_{ki}$$

and, using taking account of $K_{kjy}{}^{x}\xi^{y}=0$ and (2.13), (ii),

(2.16) $$K_{kjy}{}^{x}=K_{kji}{}^{h}f_{y}^{i}f_{h}^{x}+f_{yk}f_{j}^{x}-f_{yj}f_{k}^{x}.$$

Equation (2.15) and (2.16) mean that M^{n} is of constant curvature 1 if and only if the connection induced in the normal bundle is of zero curvature.

3. Proofs of the main theorems

We first of all remember that the equations of Gauss, Codazzi and Ricci are respectively

(3.1) $$\begin{cases} K_{kjih}=K_{\nu\mu\lambda\kappa}B_{kjih}^{\nu\mu\lambda\kappa}+h_{khx}h_{ji}{}^{x}-h_{jhx}h_{ki}{}^{x}, \\ 0=K_{\nu\mu\lambda\kappa}B_{kji}^{\nu\mu\lambda}C_{y}^{\kappa}-(\nabla_{k}h_{jiy}-\nabla_{j}h_{kiy}), \\ K_{kjyx}=K_{\nu\mu\lambda\kappa}B_{kj}^{\nu\mu}C_{yx}^{\lambda\kappa}-(h_{k}{}^{t}{}_{y}h_{jtx}-h_{j}{}^{t}{}_{y}h_{ktx}), \end{cases}$$

where $K_{\nu\mu\lambda\kappa}$, K_{kjih} and K_{kjyx} are covariant components of the curvature tensors of M^{2m+1}, M^{n} and the normal bundle respectively,

$$B_{kjih}^{\nu\mu\lambda\kappa}=B_{\kappa}^{\nu}B_{j}^{\mu}B_{i}^{\lambda}B_{h}^{\kappa} \text{ and } B_{kji}^{\nu\mu\lambda}=B_{k}^{\nu}B_{j}^{\mu}B_{i}^{\lambda}.$$

We assume that the C-Bochner curvature tensor of M^{2m+1} vanishes identically. Then from (1.6), we have

(3.2) $$\begin{aligned} &K_{\nu\mu\lambda\kappa}+(g_{\nu\kappa}-\xi_{\nu}\xi_{\kappa})L_{\mu\lambda}-(g_{\mu\kappa}-\xi_{\mu}\xi_{\kappa})L_{\nu\lambda} \\ &\quad+L_{\nu\kappa}(g_{\mu\lambda}-\xi_{\mu}\xi_{\lambda})-L_{\mu\kappa}(g_{\nu\lambda}-\xi_{\nu}\xi_{\lambda}) \\ &\quad+\phi_{\nu\kappa}M_{\mu\lambda}-\phi_{\mu\kappa}M_{\nu\lambda}+M_{\nu\kappa}\phi_{\mu\lambda}-M_{\mu\kappa}\phi_{\nu\lambda}-2(\phi_{\nu\mu}M_{\lambda\kappa}+M_{\nu\mu}\phi_{\lambda\kappa}) \\ &\quad+(\phi_{\nu\kappa}\phi_{\mu\lambda}-\phi_{\mu\kappa}\phi_{\nu\lambda}-2\phi_{\nu\mu}\phi_{\lambda\kappa})=0, \end{aligned}$$

from which, using $g_{\mu\lambda}B_{ji}^{\mu\lambda}=g_{ji}$, $\phi_{\mu\lambda}B_{ji}^{\mu\lambda}=f_{ji}$, $\phi_{\mu\lambda}B_{j}^{\mu}C_{y}^{\lambda}=-f_{jy}$, $\phi_{\mu\lambda}C_{yx}^{\mu\lambda}=0$, $\xi_{\mu}B_{k}^{\mu}=\xi_{k}$ and $\xi_{\mu}C_{y}^{\mu}=\xi_{y}$, we find

(3.3) $$\begin{cases} K_{\nu\mu\lambda\kappa}B_{kjih}^{\nu\mu\lambda\kappa}+(g_{kh}-\xi_{k}\xi_{h})L_{ji}-(g_{jh}-\xi_{j}\xi_{h})L_{ki}+L_{kh}(g_{ji}-\xi_{j}\xi_{i}) \\ -L_{jh}(g_{ki}-\xi_{k}\xi_{i})+f_{kh}M_{ji}-f_{jh}M_{ki}+M_{kh}f_{ji}-M_{jh}f_{ki} \\ -2(f_{kj}M_{ih}+M_{kj}f_{ih})+(f_{kh}f_{ji}-f_{jh}f_{ki}-2f_{kj}f_{ih})=0, \\ K_{\nu\mu\lambda\kappa}B_{kj}^{\nu\mu}C_{yx}^{\lambda\kappa}-\xi_{k}\xi_{x}L_{jy}+\xi_{j}\xi_{x}L_{ky}-\xi_{j}\xi_{y}L_{kx}+\xi_{k}\xi_{y}L_{jx}-f_{kx}M_{jy} \\ +f_{jx}M_{ky}-M_{kx}f_{jy}+M_{jx}f_{ky}-2f_{kj}M_{yx}+(f_{kx}f_{jy}-f_{jx}f_{ky})=0, \end{cases}$$

where $L_{ji}=L_{\mu\lambda}B_{ji}^{\mu\lambda}$, $L_{ky}=L_{\mu\lambda}B_k^{\mu}C_y^{\lambda}$, $M_{ji}=M_{\mu\lambda}B_{ji}^{\mu\lambda}=-L_{jh}f_i^h+L_{jx}f_i^x$, $M_{ky}=M_{\mu\lambda}B_k^{\mu}C_y^{\lambda}=-L_{ki}f_y^i$ and $M_{yx}=M_{\mu\lambda}C_{yx}^{\mu\lambda}=-L_{yi}L_x^i$.

Thus equations (3.3) can respectively be written as

$$(3.4)\quad \begin{cases} K_{kjih}+(g_{kh}-\xi_k\xi_h)L_{ji}-(g_{jh}-\xi_j\xi_h)L_{ki}+L_{kh}(g_{ji}-\xi_j\xi_i) \\ -L_{jh}(g_{ki}-\xi_k\xi_i)+f_{kh}M_{ji}-f_{jh}M_{ki}+M_{kh}f_{ji}-M_{jh}f_{ki} \\ -2(f_{kj}M_{ih}+M_{kj}f_{ih})+(f_{kh}f_{ji}-f_{jh}f_{ki}-2f_{ki}f_{ih}) \\ -(h_{khx}h_{ji}{}^x-h_{jhx}h_{ki}{}^x)=0, \\ K_{kjyx}-(\xi_kL_{jy}-\xi_jL_{ky})\xi_x-(\xi_jL_{kx}-\xi_kL_{jx})\xi_y-f_{kx}M_{jy}+f_{jx}M_{ky} \\ -M_{kx}f_{jy}+M_{jx}f_{ky}-2f_{kj}M_{yx}+(f_{kx}f_{jy}-f_{jx}f_{ky}) \\ +(h_k{}^t{}_yh_{jtx}-h_j{}^t{}_yh_{ktx})=0. \end{cases}$$

I. *The case in which the vector field ξ^κ is tangent to M^n.*

We now consider the case in which the vector field ξ^κ is tangent to the anti-holomorphic submanifold M^n, that is, $\xi^\kappa=0$. When $n=m+1$, we can easily find $f_{ji}=0$. Thus the second equation of (3.4) becomes

$$K_{kjyx}-f_{kx}M_{jy}+f_{jx}M_{ky}-M_{kx}f_{jy}+M_{jx}f_{ky}$$
$$+(f_{kx}f_{jy}-f_{jx}f_{ky})+(h_k{}^t{}_yh_{jtx}-h_j{}^t{}_yh_{ktx})=0,$$

from which, transvecting with $f_i^y f_h^x$ and using $f_j^x f_{ix}=g_{ji}-\xi_j\xi_i$ derived from (2.9), (i), we find

$$(3.5)\quad K_{kjyx}f_i^yf_h^x-(g_{kh}-\xi_k\xi_h)M_{jy}f_i^y+(g_{jh}-\xi_j\xi_h)M_{ky}f_i^y$$
$$-M_{kx}f_h^x(g_{ji}-\xi_j\xi_i)+M_{jx}f_h^x(g_{ki}-\xi_k\xi_i)+(g_{kh}-\xi_k\xi_h)(g_{ji}-\xi_j\xi_i)$$
$$-(g_{jh}-\xi_j\xi_h)(g_{ki}-\xi_k\xi_i)+(h_k{}^t{}_yh_{jtx}-h_j{}^t{}_yh_{ktx})f_i^yf_h^x=0.$$

We now assume that the second fundamental tensors are commutative. Then from (2.12) and (3.5), we have

$$(3.6)\quad K_{kjih}+(g_{kh}-\xi_k\xi_h)N_{ji}-(g_{jh}-\xi_j\xi_h)N_{ki}+N_{kh}(g_{ji}-\xi_j\xi_i)$$
$$-N_{jh}(g_{ki}-\xi_k\xi_i)+(g_{kh}-\xi_k\xi_h)(g_{ji}-\xi_j\xi_i)-(g_{jh}-\xi_j\xi_h)(g_{ki}-\xi_k\xi_i)=0,$$

where $N_{ji}=-M_{jy}f_i^y$.

Now since the vector field ξ^h is parallel, the Riemannian manifold M^n is locally a product M^1 generated by ξ^h and M^{n-1} totally geodesic in M^n. We represent M^{n-1} in M^n by parametric equations $y^h = y^h(z^a)$ $(a, b, c, \cdots = 1, 2, \cdots, (n-1))$ and put $B_b^h = \partial y^h / \partial z^b$. Then we have $\xi_i B_b^i = 0$ and the curvature tensor K_{dcba} of M^{n-1} is given by

$$K_{dcba} = K_{kjih} B_{dcba}^{kjih}, \quad \text{where } B_{dcba}^{kjih} = B_d^k B_c^j B_b^i B_a^h. \tag{3.7}$$

Thus transvecting (3.6) with B_{dcba}^{kjih}, we obtain

$$K_{dcba} + g_{da} C_{cb} - g_{ca} C_{db} + C_{da} g_{cb} - C_{ca} g_{db} = 0, \tag{3.8}$$

where $g_{cb} = g_{ji} B_c^j B_b^i$ is the metric tensor of M^{n-1} and

$$C_{cb} = N_{ji} B_c^j B_b^i + \frac{1}{2} g_{cb}.$$

Equation (3.8) shows that the Weyl conformal curvature tensor of M^{n-1} vanishes and M^{n-1} is conformally flat if $n-1 \geqq 4$. Thus we have completely proved Theorem 1.

II. *The case in which the vector field ξ^κ is normal to M^n*

We now assume that $n = m$. Then the vector field ξ^κ is normal to M^n and $f_{ji} = 0$. Then from the first equation of (3.4) we have

$$K_{kjih} + g_{kh} L_{ji} - g_{jh} L_{ki} + L_{kh} g_{ji} - L_{jh} g_{ki} - (h_{khx} h_{ji}{}^x - h_{jhx} h_{ki}{}^x) = 0. \tag{3.9}$$

If M^n is umbilical, that is, if $h_{ji}{}^x = g_{ji} h^x$, then we can write (3.9) in the form

$$K_{kjih} + g_{kh} D_{ji} - g_{jh} D_{ki} + g_{ji} D_{kh} - g_{ki} D_{jh} = 0, \tag{3.10}$$

where $D_{ji} = L_{ji} - \frac{1}{2} h_x h^x g_{ji}$.

Equation (3.10) show that the Weyl conformal curvature tensor of M^n vanishes. Thus we have completely proved Theorem 2.

We next obtain from the second equation of (3.4)

$$\begin{aligned} K_{kjyx} &+ M_{ky} f_{jx} - M_{jy} f_{kx} + f_{ky} M_{jx} - f_{jy} M_{kx} \\ &+ (f_{kx} f_{jy} - f_{jx} f_{ky}) + (h_k{}^t{}_y h_{jtx} - h_j{}^t{}_y h_{ktx}) = 0. \end{aligned} \tag{3.11}$$

If the second fundamental tensors of M^n commute, then we have from (3.11)

$$K_{kjyx} - f_{kx} M_{jy} + f_{jx} M_{ky} - M_{kx} f_{jy} + M_{jx} f_{ky} + (f_{kx} f_{jy} - f_{jx} f_{ky}) = 0, \tag{3.12}$$

from which, by transvecting with $f_i^y f_h^x$ and using (2.13), (i)

$$K_{kjyx} f_i^y f_h^x - g_{kh} M_{jy} f_i^y + g_{jh} M_{ky} f_i^y - M_{ky} f_h^y g_{ji} + M_{jy} f_h^y g_{ki} + (g_{kh} g_{ji} - g_{jh} g_{ki}) = 0. \tag{3.13}$$

Substituting (3.13) into (2.15), we find

$$K_{kjih} - g_{kh} M_{jy} f_i^y + g_{jh} M_{ky} f_i^y - M_{ky} f_h^y g_{ji} + M_{jy} f_h^y g_{ki} = 0, \tag{3.14}$$

which shows that the Weyl conformal curvature tensor of M^n vanishes. Thus we have completely proved Theorem 3.

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