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ON THE CHARACTERIZATIONS OF CERTAIN RADICAL CLASSES

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All rings in this paper are assumed to be associative and the major knowledge of radical theory required for our purposes is contained in [4]. In [5] le-Roux and Heyman introduced the concept of an *h_M-ring* as follows:

DEFINITION 1. Let M be an arbitrary class of rings.

(a) A non-simple ring R is called an *h_M-ring* if:

(i) $R/I \in M$ for every nonzero ideal I of R.

(ii) Every minimal ideal of R belongs to M.

(b) A simple ring R is an *h_M-ring* if and only if $R \in M$.

The class of all h_M-rings is denoted by M^*. We assume that the ring 0 belongs to every non-void class of rings.

This definition enables us to characterize certain radical classes and the purpose of this note is to present characterizations of three well-known ones.

1. The Behrens radical class

The Behrens radical J_B [3] is the upper radical determined by the class of subdirectly irreducible rings such that heart of each ring contains nonzero idempotent elements. In [6] Propes offered a new characterization for J_B as a lower radical class, namely $J_B = P$ where $P = \{R \mid R$ has no homomorphic image with nonzero idempotent elements$\}$. If we denote by M the class of all rings without nonzero idempotent elements and define M^* as above, the results of Propes become easy consequences of our considerations.

First we need

LEMMA 1. *For any hereditary class C of rings, C^* is homomorphically closed.*

PROOF Let $R \in C^*$. The case where R is simple is trivial. Let then I be any nonzero ideal of R, so definition 1 implies that $R/I \in C$. R/I simple, implies $R/I \in C^*$ according to definition 1. If R/I is not simple, let K/I be any non trivial ideal of R/I. Then $(R/I)/(K/I) \cong R/K$. Since K is a nonzero ideal of $R \in C^*$ it follows that $R/K \in C$ and therefore $(R/I)/(K/I) \in C$. If R/I contains a minimal ideal S/I, then $S/I \in C$ since C is hereditary. This proves the lemma.

Since our class M is obviously hereditary, lemma 1 implies that M^* is homomorphically closed.

THEOREM 1. $J_B = M^*$ (See also [6]).

PROOF. Suppose $R \in M^*$ can be mapped homomorphically onto a nonzero subdirectly irreducible ring R/J with heart H/J which contains nonzero idempotent elements. Since M^* is homomorphically closed it follows that $R/J \in M^*$. The heart H/J is a minimal ideal of R/J and hence $H/J \in M$. This is in contradiction with the construction of M. Hence $M^* \subset J_B$. Conversely let $R \in J_B$. Since R does not contain nonzero idempotent elements ([3], theorem 7) and J_B is homomorphically closed, no homomorphic image of R can contain nonzero elements. Thus $R \in M^*$ which implies $J_B \subset M^*$. The proof of the theorem is completed.

2. The antisimple radical

Andrunakievič's antisimple radical class β_ϕ (see [1]) is the upper radical class determined by the class of all subdirectly irreducible rings with idempotent hearts. Amongst other results in his paper [2], Andrunakievič proved the following

LEMMA 2. ([2], theorem 3). *The ring $R \in \beta_\phi$ if and only if for every homomorphic image $\bar{R}$ of R we have $(\bar{a})^2 \neq (\bar{a})$ for every nonzero principal ideal $(\bar{a})$ of $\bar{R}$.*

In order to give a characterization of β_ϕ as a lower radical class we give

DEFINITION 2. A nonzero element a of a ring R is called an *α-element of R* if $(a) \neq (a)^2$ where (a) is the principal ideal generated by a in R. R is called an *α-ring* if every nonzero element of R is an α-element.

If we denote the class of all α-rings by A, we obtain

THEOREM 2. $A^* = \beta_\phi$.

PROOF. Let $R \in A^*$. If R is simple. we have $R \in A$.

Then, in view of the fact that $(a) \neq (a)^2$ for any $0 \neq a \in R$, it follows that R is a zero-ring and hence $R \in \beta_\phi$. If R is a non-simple ring, let R/J be any nonzero homomorphic image of R. Since $R \in A^*$ it follows that $R/J \in A$. For every $0 \neq \bar{a} \in R/J$ it follows that $(\bar{a}) \neq (\bar{a})^2$ and by lemma 2 we therefore have

$A^* \subset \beta_\phi$.

Conversely, if $R \in \beta_\phi$ the construction of A^* and lemma 2 imply that $R \in A^*$. Hence $\beta_\phi \subset A^*$ so that the theorem is proved.

3. The radical class determined by the class of almost nilpotent rings

In their paper [7] Van Leeuwen and Heyman introduced and studied almost nilpotent rings where a ring R is defined to be almost nilpotent if every nonzero ideal of R strictly contains a power of R. Following Van Leeuwen and Heyman we will denote by L_1 the class of all almost nilpotent rings. Denoting by H the class of all nilpotent rings, it can readily be verified that $H \subset H^*$. The inclusion is strict in view of the following example.

EXAMPLE 1. Let $W = \left\{ \frac{2x}{2y+1} \,\middle|\, (2x, 2y+1) = 1,\ x, y \in Z \right\}$, ([4], p. 103). The only nonzero ideals of W are of the type $(2)^n$, $n = 1, 2, 3, \cdots$, and $W = (2)$. For any nonzero ideal I of W it follows that $W/I \in H$. Note that W has no minimal ideals and therefore $W \in H^*$ although $W \notin H$.

We offer another example to illustrate that H* fails to be a radical class.

EXAMPLE 2. Consider the *Zassenhaus-ring* A consisting of all finite sums $\Sigma a_\alpha x_\alpha$ where α is a rational with $0 < \alpha < 1$, the a_α-s being elements of the two element field Z_2 and the x_α-s are indeterminates such that

$$x_\alpha x_\beta = \begin{cases} x_{\alpha+\beta} & \text{if } \alpha+\beta < 1 \\ 0 & \text{if } \alpha+\beta \geq 1 \end{cases}$$

(cf. [4], p. 19).

A as a ring, is a nil ring and $A = A^2$. Furthermore $A/I \notin H$ for every nontrivial ideal I of A, which implies that $A \notin H^*$. The principal ideal (x_α) generated by any basis element x_α is nilpotent since $(x_\alpha)^n = 0$ for any $n > \frac{1}{\alpha}$. Every nonzero homomorphic image of A contains a nonzero nilpotent ideal and consequently an H^*-ideal. It follows therefore that $A \in LH^*$ where LH^* is the lower radical class determined by H^*. Hence $LH^* \neq H^*$ so that H^* is no radical class.

In order to characterize L_1 we present

THEOREM 3. *A ring R is almost nilpotent if and only if $R \in H^*$.*

PROOF. Suppose R is any almost nilpotent ring. Then $R^n \subset I$ for any nonzero ideal I of R and some $n \in N$. If R is a simple ring, it is easily verified that $R \in H \subset H^*$. If R is not a simple ring it follows that $R/I \in H$ for every nonzero ideal I of R since $R^n \subset I$. Next, if M is a minimal ideal of R, then since $R^m \subset M$ for a certain $m \in N$, we have that $R^m = 0$. This means that R, and hence every ideal of R is nilpotent. In particular $M \in H$. We conclude that R satisfies all the requirements of definition 1 and therefore $R \in H^*$, that is $L_1 \subset H^*$. Conversely, let $R \in H^*$. If R is simple, then $R \in H$ and therefore $R \in L_1$. If R is not simple, let I be any nonzero ideal of R. By the definition of H^*, this implies that $R/I \in H$. Hence $R^m \subset I$ for some $m \in N$. Suppose there does not exist a $k \in N$ such that $R^k \subset I$, hence $R^k = I$ for all $k \geq m$. If I is not minimal in R, there exists a nonzero ideal J of R with $J \subset I$. Since $R \in H^*$ it follows that $R/J \in H$. Hence $R^s \subset J$ for some $s \in N$ and $s \geq m_0$. Then however $R^s \subset I$, $s > max\ (m, m_0)$, which contradicts the fact that $R^k = I$ for all $k \geq m$. We may therefore assume that I is a minimal ideal of R. According to definition 1, it follows that $I \in H$ which again contradicts the fact that $R^k = I$ for all $k \geq m$. Consequently there exists an $m \in N$ such that $R^m \subset I$. R is therefore almost nilpotent and the reverse inclusion is established.

Therefore we obtain

COROLLARY 1. $LL_1 = LH^*$.

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