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COMPACTNESS IN PAIRWISE SKOROKHOD CONVERGENT TOPOLOGY

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ABSTRACT

近來 Skorokhod는 確率論의 極限問題와 關聯하여 모든 不連續函數空間에 關한 位相을 定義하였다.

本 論文에서는 Skorokhod 收斂位相을 雙位相 (bitopology)型으로 一般化하고 잘 알려져 있는 여러位相과 比較하여 다음과 같은 結果를 새로 얻었다.

(定理 2-11); 空間 X와 Y가 完備準距離可分空間 (Completely quasi—metric separable space)이라면 雙概收斂位相(pairwise almost convergent topology)는 Skorokhod 双收斂位相 보다 弱하다. 그리고

(定理 2—12); 雙 graph 位相은 Skorokhod J_1-收斂位相과 一致한다.

끝으로 主定理인 (定理 3—1)과 (定理 3—2)에서 Skorokhod 雙收斂位相의 Compact性에 關한 必要充分條件을 밝혔다.

I. INTRODUCTION

The study on the function space topologies has mainly been investigated in the space of continuous function (1)

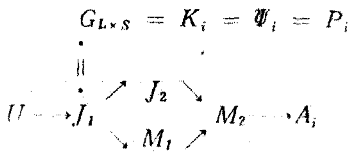
Recently, Skorokhod (2) defined new topologies on the space of all discontinuous of the first kind connection with a problem in probability theory.

In this paper Skorokhod convergent topologies are generalized by the form of the bitopological space and compared with the other known topologies. (3) (5). That is,

(2-11 theorem): If X and Y are completely quasi metric separable, then pairwise almost convergent topology is coarser than pairwise Skorokhod convergent topology.

(2-12 theorem): The bigraph topology coincides with Skorokhod J_1 convergent topology. (6)

(2 13 corollary): Therefore denoting the statement "the topology S_1 is stronger than S_2" by $S_1 \longrightarrow S_2$' we have the following relation:



(3-1 Theorem) and (3-2 Theorem): Finally, We show necessary and sufficient condition for compactness in pairwise Skorokhod convergent topology. (4)

2. Pairwise Skorokhod convergent topology

2-1 DEFINITION: Let (X, d_1, d_2) and (Y, d^*_1, d^*_2) be completely quasi metric separable spaces where $d_2(d_2^*)$ is a conjugate metric of $d_1(d_1^*)$.

We define the metric R_i by

$$R_i((x_1, y_1), (x_2, y_2)) = d_i^*(y_1, y_2) + d_i(x_1, x_2), \quad i = 1, 2.$$

Then we obtain pairwise Skorokhod convergent topological space (Y^X, R_1, R_2).

2-2 DEFINITION: We denote by $K(X, Y)$ the space of all functions $f(x)$ which are defined on the interval $X = [0, 1]$, whose values lie in X, and which are every point have a limit on the left and continuous on the right (and on the left at t=1).

Let us consider certain properties of the functions which belong to $K(X, Y)$. A function $f(x)$ will be said to have a discontinuity

$$d^*_i(f(x_0 - 0), f(x_0 + 0)), \quad i = 1, 2$$

at the point x_0.

2-3 Lemma: If $f(x) \in K(X, Y)$, then for any positive ε there exists only a finite number of values of x such that the discotinuity of $f(x)$ is greater than ε.

PROOF: This follows from the fact that if there exists a sequence for which $x_k \longrightarrow x_0$ such that

$$d_i^*(f(x_k + 0), f(x_k - 0)) > \varepsilon, \quad i = 1, 2$$

then at x_0 the function $f(x)$ would have no limit either on the right or on the left.

2-4 Lemma: Let $x_1, x_2, x_3, \cdots\cdots x_k$ be all the points at which $f(x)$ has discontinuities no less than ε. Then there exists a δ such that if $[x' - x''] < \delta$ and if x' and x'' both being to the same one of the intervals $(0, k_1)$, $(x_1, x_2), \cdots\cdots$ (x_2, x_3) $(x_k, 1)$, then

$$d_i^*(f(x'), f(x'')) < \varepsilon, \quad i = 1, 2.$$

PROOF: Assume the contrary. Then there would exist sequences x'_n and x''_n which converge to some point x_0 and belong to the same one of the intervals $(0, x_1)$, $(x_k, 1)$ and the sequences would have the property that

$$d^*_i(f(x'_n),\ f(x''_n)) \geq \varepsilon,\ i=1,2.$$

Now the points x'_n and x''_n lie on opposite sides of x_0 (otherwise $d^*_i(f(x'_n),\ f(x''_n)) \geq \varepsilon$ would be impossible), so that $d^*_i(f_0 x+0),\ f(x_0-0)) \geq \varepsilon,\ i=1,2$. Therefore x_0 is one of the points $x_1, x_2, \cdots\cdots x_k$, which contradicts the statement that x'_n and x''_n belong to the same one of the intervals $(0, x_1)$, $(x_1, x_2), \cdots\cdots (x_k, 1)$.

2-5 DEFINITION: The sequence of functions $f_n(x)$ converges uniformly to $f(x)$ at the point x_0 if for all $\varepsilon > 0$ there exists a δ such that

$$\overline{\lim_{n\to\infty}} \sup_{|x-x_0|<\delta} d^*_i(f_n(x),\ f(x_0)) < \varepsilon,\ i=1,2.$$

Obviously if $f_n(x)$ converges uniformly to $f(x)$ at every point of some closed set, then $f_n(x)$ converges uniformly to $f(x)$ on this whole set.

2-6 DEFINITION: The sequence $f_n(x)$ is called J_1 convergent to $f(x)$ if there exist a sequence of continuous one to one mappings $\lambda_n(S)$ of the interval $X=[0,1]$ onto itself, such that

$$\lim_{n\to\infty} \sup_s R_i[(s, f_n(s)), (\lambda_n(s), f(\lambda_n(s)))]=0,\ i=1,2.$$

The uniform convergent topology U and J_1 convergent topology J_1 take the form of a single jump at a discontinuity point x_0. In both these topologies, for values of x close to x_0, the function $f_n(x)$ can take on values which are either close to $f(x_0-0)$ or to $f(x_0+0)$.

If we wish to keep this last property, but do not require that the transition be in the form of a single jump, that is, that a function $f_n(x)$ may change back and forth between the values $f(x_0-0)$ and $f(x_0+0)$ several times in the neighborhood of a point x_0, then we obtain topology J_2.

2-7 DEFINITION: A sequence $f_n(x)$ is said to be J_2 convergent to $f(x)$ if there exists a sequence of one to one mapping $\lambda_n(x)$ of the interval $X=[0,1]$ onto itselt such that

$$\lim_{n\to\infty} \sup_s R_i[(s, f_n(s)),\ (\lambda_n(s),\ f(\lambda_n(s)))]\to 0,\ i=1,2.$$

2-8 DEFINITION: The pair of of functions $(x(s), f(s))$ gives a parametric representaion of the graph (x, y) if those and only those pairs (x, y) belong to it for which an s can be found such that $y=f(s)$, where $f(s)$ is continuous, and $x(s)$ is continuous and monotonically increasing (the functions $f(s)$ and $x(s)$ are defined on the segment $[0,1]$).

We note that if $(f_1(s),\ x_1(s))$ and $(f_2(s),\ x_2(s))$ are parametric representations of $x(s)$, there exists a monotonically increasing function $\lambda(s)$ such that

$$f_1(s)=f_2(\lambda(s)) \text{ and } x_1(s)=x_2(\lambda(s)).$$

2-9 DEFINITION: The sequence $f_n(x)$ is called M_1-convergent to $f(x)$ if there exist parametric representations $(x(s),\ f(s))$ of $f(x)$ and $(x_n(s),\ f_n(s))$ of $f_n(x)$ such that

$$\lim_{n\to\infty} \sup_{s} R_i[(x_n(s),\ f_n(s)),\ (x(s),\ f(s))]=0,\quad i=1,2.$$

We can characterize the topology M_1 in the following way from the point of view of the behavior at a point of discontinuity x_0 of the function $f(x)$.

The transition from $f(x_0-0)$ to $f(x_0+0)$ is such that first $(f_n(x)$ is arbitrarily close to the segment $[f(x_0-0),\ f(x_0)]$ and second that $f_n(x)$ moves from $f(x_0-0)$ to $f(x_0)$ almost always advancing.

2-10 DEFINITION: The sequence $f_n(x)$ is called M_2-convergent to $f(x)$ if

$$\lim_{n\to\infty} \sup_{(x_1,y_1)\in C(f(x))} \inf_{(x_2,y_2)\in C(f_n(x))} R_i[(x_1,y_1),\ (x_2,y_2)]=0,\quad i=1,2.$$

Let G be any of our topologies. We shall denote convergence in the topology G by the symbol

$$f_n(x)\xrightarrow{G} f(x).$$

Let us consider the relation between our topologies. It is clear that U is stronger than J_1, and that this in turn is stronger than J_2. It is also clear that M_1 is stronger than M_2. We recall that a topology G_1 is stronger than G_2 if convergence in G_1 implies convergence in G_2. If X is a linear space, we can use any of our topologies. It is easily seen that convergence in M_2 follows from convergence in any of the other topologies, and that convergence in J_1 implies convergence in any of other topologies except ordinary uniform convergent topology U.

Denoting the statement "the topology G_1 is stronger than G_2" by $G_1 \longrightarrow G_2$, all the above can be summarized by

$$U \longrightarrow J_1 \begin{matrix} \nearrow M_1 \searrow \\ \searrow J_2 \nearrow \end{matrix} M_2$$

(Skorokhod(2))

2.11 THEOREM: p-almost convergent topology is coarser than p-M_2-convergent topology.

PROOF: Let (X, L_1, L_2) and (Y, S_1, S_2) be bitopological space and we define

$$A_i(U,V)=\{f\in Y^X:\ f(U)\cap V\neq\phi)$$

where $U\in L_i$ and $V\in S_i, i=1,2.$
If T_i is generated by $\{A_i(U,V)\}$, (Y^X, T_1, T_2) is said to be p-almost convergent topology.
Since M_2-metric is

$$R_i((x_1,y_1),\ (x_2,y_2))=d_i^*(x_1,y_2),+d_i(x_1,y_2)$$

If we take

$$\varepsilon_0=\frac{1}{2}\ d_i^*(y_1,y_2),\ \ \delta_0=\frac{1}{2}\ d_i(x_1,x_2),\ \ x_0 \text{ and } y_0$$

such that

$$x_0 = \inf\{x \mid d_i(x_1,x) = d_i(x,x_2)\}$$

and

$$y_0 = \inf\ \{y \mid d_i^*(y_1,y) = d_i^*(y,y_2)\},$$

we can express

$$U=s_{d_i}(x_0,\delta_0)=\{z:d_i(x_0,z) <\delta_0\} \text{ and } V=S_{d_i}^*(y_0,\varepsilon_0)$$
$$=\{z:\ d_i^*(y_0,z) <\varepsilon_0).$$

And so, we can make an open sphere $S_{R_i}(f,\varepsilon_j)$ such that

$$A_j(U,V) = S_{R_i}(f,\varepsilon_j) = \{f\in Y_X:\ f(x)\in V \text{ for any } x\in U\}.$$

2.12 THEOREM: Lex X and Y are completely quasi-metric separable. The bigraph topology conicides with the pairwise J_1-convergent topology.
PROOF: Let

$$G_{wi} = \{f\in Y^X:\ C(f) \subset W\in L_i\times S\},\ \ i=1,2.$$

Then G_{wi} is a subbasic open set in bigraph topology. We define

$$W=(X-\{x_0\})\times Y\cup X\times S_{d_i}^*(y_0,\varepsilon) \text{ as } \varepsilon\longrightarrow 0,$$

and $S_{R_i}(f,\varepsilon)=\{f\in Y^x:\ f(x_0)=y_0\}.$

Since the metric, in J convergent topology, is

$$\lim_{n\to\infty} \sup_{s} R_i[(s, f(s)), (\lambda_n(s), f(\lambda_n(s))] = 0, \quad i = 1, 2.$$

if $\lambda_n(s) = x_n$, $S = x$, $y = f(s)$ and $y_n = f(\lambda_n(s))$, then

$$\lim_{n\to\infty} \sup_{x} d_i^*(y, y_n) = 0 \text{ and}$$

$$\lim_{n\to\infty} \sup_{x} d_i(x, x_n) = 0.$$

Therefore $G_{wi} = S_{R_i}(f, \varepsilon)$ where $S_{R_i}(f, \varepsilon)$ is a subbasic open set in pairwise J_1-convergent topology.

2-13 COROLLARY: Let X and Y be completely quasi-metric separable. Then, by (1), (5), (2-11) and (2-12), we have the following relation:

$$G_{L\times S} = K_i = \Psi_i = P_i = J_1 \supset M_2 \supset A_i$$

i.e.

denothing the statement "the topology S_1 is stronger than S_2" by $S_1 \longrightarrow S_2$, all the above can be summarized by

$$G_{L\times S} = K_i = \Psi_i = P_i$$

$$\|$$

$$U \longrightarrow J_1 \begin{matrix} \nearrow J_2 \searrow \\ \searrow M_1 \nearrow \end{matrix} M_2 \longrightarrow A_i$$

3. Compactness conditions in pairwise Skorokhod convergent topology.

3-1 THEOREM: (necessary condition) If a set of functions $K \subset K(X, Y)$ is to be compact in one of the topologies J_1, J_2, M_1 and M_2, it is necessary that for all $x \in X$ and $f(x) \in K(X, Y)$ the values of $f(x)$ belong to a single compact set A of X.

PROOF: If we have a sequence of points $f_n(x_n)$, then by choosing sequence n_k such that

$$f_n(x) \xrightarrow{M_2} f_0(x), \quad x_{nk} \longrightarrow x_0,$$

we find that the distance between $f_{nk}(x_{nk})$ and the segment $[f_0(x_0 - 0), f_0(x_0)]$ approaches zero, which means that $f_{nk}(x_{nk})$ is comact, so that the segment is compact.

3-2 THEOREM: (sufficient condition) The set of function K is compact in a topology S, where S is J_1, J_2, M_1 or M_2, if (3-1 Theorem) is fulfilled and if

$$\lim_{c\to 0} \lim_{f(x)\in K} \sup \left(\triangle_s (c, f(x)) + \sup_{0<x<c} d^*_i(f(o), f(x)) + \sup_{1-c<x<1} d^*_1(f(1), f(x))\right)) = 0, \ i=1,2.$$

Where $\triangle_s(\varepsilon, f(x)) = \sup_{x\in[0,1], x_1\in[x_0, x_0+\frac{\varepsilon}{2}], x_2\in[x'_0-\frac{\varepsilon}{2}, x'_0]} H(fx_1), f(x), f(x_2))$

$x_0 = max\ [0, x-\varepsilon]$, $x'_0 = min[x, x+\varepsilon]$ and $H(f(x_1), f(x), f(x_2))$ is the distance of $f(x)$ from the segment $[f(x_1), f(x_2)]$

(by lim sup of a certain numerical set we mean its maximum limit point).

PROOF: Choosing everywhere dense set N of values of values of x which contain 0 and 1, we may take from any sequence $f_n(x)$ a subsequence $fn_k(x)$ such that if $x\in N$ then

$$\lim_{n_k\to\infty} fn_k(x)$$

exists. Since

$$\triangle_{M2} (c, f(x) \leqq \triangle_s (c, f(x)),$$

we have

$$\lim_{c\to 0} \overline{\lim_{n_k\to\infty}} \left(\triangle_{M2} (c, fn_k(x)) + \sup_{o<x<c} d^*_i(f(0), f(x)) + \sup_{1-c<x<1} d^*_i(f(1)` f(x)\right) = 0.$$

According to (2), therefore, there exists a $\bar{f}(x)\in k(X,Y)$ such that

$$fn_k(x) \xrightarrow{M_2} \bar{f}(x).$$

This means that $fn_k(x)$ converges to $\bar{f}(x)$ on everywhere dense set containing 0 and 1, and that

$$\lim_{c\to 0} \lim_{f(x)\in k} \sup(\triangle_s(c, f(x)) + \sup_{o<x<c} d_i^* (f(o), f(x))) + \sup_{1-c<x<1} d^*_i(f(1), f(x)) = 0, \ i=1,2,$$

is fulfilled.

Hence $\qquad fn_k(x) \xrightarrow{s} \bar{f}(x).$

Bearing in mind that $J_1(c, f(x))$ and $M_1(c, f(x))$ are monotonic function of c, it is easy to obtain the following:

3-3 COROLLARY: (Necessary and sufficient condition for compactness in the topologies J_1, M_1)

Let S_1 denote either J_1 or M_1. Conditions (3-1 Theorem) and

$$\lim_{c\to 0}\ \sup_{f(x)\in K}\ (\triangle_{S_1}(c,\ f(x)) + \sup_{0<x<c} d^*_i(f(o),\ f(x)) +$$

$$\sup_{1-c<x<1} d^*_i(f(1),\ f(x))) = 0,\ i=1,2,$$

are necessary and sufficient for K to be compact in S_1.

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