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ON A SPECIAL *H*-PROJECTIVE CHANGE IN A KAEHLERIAN MANIFOLD

BY UN KYU KIM AND SANG-SEUP EUM

§ 0. Introduction.

In an almost complex manifold with structure tensor $\varphi_i{}^h$, an affine connection Γ is called an *φ-connection* if the almost complex structure φ is covariantly constant with respect to this connection.

In a complex manifold with a symmetric φ-connection, we consider a curve $\xi^h(t)$ satisfying differential equations

$$\frac{d^2\xi^h}{dt^2} + \Gamma_{ji}{}^h(\xi)\frac{d\xi^j}{dt}\frac{d\xi^i}{dt} = \alpha(t)\frac{d\xi^h}{dt} + \beta(t)\varphi_j{}^h\frac{d\xi^j}{dt},$$

where $\alpha(t)$ and $\beta(t)$ are certain functions of the parameter t. We call such a curve *a holomorphically planar curve*. If two symmetric φ-connection Γ and $'\Gamma$ have all the holomorphically planar curves in common, they are said to be *H-projective related* to each other.

It is known [1] that two symmetric φ-connections Γ and $'\Gamma$ are H-projectively related to each other when and only when

$$'\Gamma_{ji}{}^h = \Gamma_{ji}{}^h + \delta_j{}^h p_i + \delta_i{}^h p_j + \varphi_j{}^h q_i + \varphi_i{}^h q_j \tag{0.1}$$

holds for a certain covector field p_i where

$$q_i = -p_t\varphi_i{}^t. \tag{0.2}$$

We call such a change of Γ an *H-projective change* of symmetric φ-connections.

It is also known that [1] the curvature tensor $'K_{kji}{}^h$ formed with $'\Gamma_{ji}{}^h$ and the curvature tensor $K_{kji}{}^h$ formed with $\Gamma_{ji}{}^h$ are related to each other by

$$\begin{aligned} 'K_{kji}{}^h = K_{kji}{}^h &+ \delta_j{}^h P_{ki} - \delta_k{}^h P_{ji} + \delta_i{}^h (P_{kj} - P_{jk}) \\ &+ \varphi_j{}^h Q_{ki} - \varphi_k{}^h Q_{ji} + \varphi_i{}^h (Q_{kj} - Q_{jk}), \end{aligned} \tag{0.3}$$

where

$$P_{ji} = \nabla_j p_i - p_j p_i + p_l p_k \varphi_j{}^l \varphi_i{}^k, \tag{0.4}$$

and

(0.5) $$Q_{ji}=-P_{jt}\varphi_i{}^t,$$

∇_j denoting the covariant differentiation with respect to the φ-connection Γ_{ji}^h.

In the present paper, we assume that there is a scalar function p such that $p_i=\partial_i p$ in a Kaehlerian manifold and we prove that the Bochner curvature tensor is invariant under such an H-projective change in a Kaehlerian manifold.

§1. Preliminaries

We consider a $2n$-dimensional Kaehlerian manifold M covered by a system of coordinate neighborhoods $\{U;\xi^h\}$, where here and in the sequel the indices $h,i,j,\cdots$ run over the range $\{1,2,\cdots,2n\}$, and denote by g_{ji} and $\varphi_j{}^h$ the components of the Hermitian metric tensor and those of the complex structure of M, respectively.

In the present paper we assume that there exist H-projectively related two symmetric φ-connections Γ_{ji}^h and $'\Gamma_{ji}^h$ in a Kaehlerian manifold M.

We denote by ∇_j the operator of covariant differentiation with respect to the symmetric φ-connection Γ_{ji}^h. We have by the help of Ricci identity,

$$K_{hkj}{}^t\varphi_t{}^i-K_{hkt}{}^i=0,$$

from which,

(1.1) $$K_{hkj}{}^t\varphi_{ti}=K_{hki}{}^t\varphi_{tj},$$

where $K_{hkj}{}^t$ is the curvature tensor formed with Γ_{ji}^h and $\varphi_{ji}=\varphi_j{}^tg_{ti}$.

Similarly, from a symmetric φ-connection $'\Gamma_{ji}^h$, we obtain

(1.2) $$'K_{hkj}{}^t\varphi_{ti}={}'K_{hki}{}^t\varphi_{tj},$$

where $'K_{hkj}{}^t$ is the curvature tensor formed with $'\Gamma_{ji}^h$.

§2. Main theorem and it's proof

In a Kaehlerian manifold $M(\varphi_j{}^h, g_{ji})$, if two φ-connections Γ_{ji}^h and $'\Gamma_{ji}^h$ are related by (0.1), then we call such a change of Γ_{ji}^h an H-projective change. If there exists a scalar function p such that $p_i=\partial_i p$, then we shall call such a change of Γ_{ji}^h an *special H-projective change*.

We now consider the so-called Bochner curvature tensor formed with Γ_{ji}^h [2] defined by

(2.1) $$B_{kji}{}^h=K_{kji}{}^h+\delta_k^hL_{ji}-\delta_j^hL_{ki}+L_k{}^hg_{ji}-L_j{}^hg_{ki}$$

$$+\varphi_k{}^h M_{ji}-\varphi_j{}^h M_{ki}+M_k{}^h\varphi_{ji}-M_j{}^h\varphi_{ki}$$
$$-2(M_{kj}\varphi_i{}^h+\varphi_{kj}M_i{}^h),$$

where

(2.2) $$L_{ji}=-\frac{1}{2(n+2)}\left[K_{ji}+\frac{1}{4(n+1)}Kg_{ji}\right],$$

(2.3) $$K_{ji}=K_{tji}{}^t,\quad K=K_{ji}g^{ji},\quad L_j{}^h=L_{jt}g^{th}$$

and

(2.4) $$M_{ji}=-L_{jt}\varphi_i{}^t,\quad M_j{}^h=M_{jt}g^{th}.$$

In this section we prove the following theorem stated at the end of the first section.

THEOREM. *The Bochner curvature tensor in a Kaehlerian manifold M (dim $M\geqq 4$) is invariant under the special H-projective change.*

Proof. We assumed that there exists a scalar function p such that $\partial_i p=p_i(\partial_i=\partial/\partial\xi^i)$. Taking account of this assumption and (0.4), we have

(2.5) $$P_{ji}=P_{ij}.$$

Substituting (2.5) into (0.3), we see that

(2.6) $$'K_{kji}{}^h=K_{kji}{}^h+\delta_j^h P_k^i-\delta_k^h P_{ji}+\varphi_j{}^h Q_{ki}-\varphi_k{}^h Q_{ji}$$
$$+\varphi_i{}^h(Q_{kj}-Q_{jk}).$$

Substituting (2.6) into (1.2) and taking account of (1.1), we have

(2.7) $$\varphi_{jh}P_{ki}-\varphi_{kh}P_{ji}-g_{jh}Q_{ki}+g_{kh}Q_{ji}$$
$$=\varphi_{ji}P_{kh}-\varphi_{ki}P_{jh}-g_{ij}Q_{kh}+g_{ki}Q_{jh}.$$

Transvecting (2.7) with g^{hk} and taking account of (0.5) and (2.5), we get

(2.8) $$(2n-1)Q_{ji}-Q_{ij}=\alpha\varphi_{ji}-g_{ji}Q_t{}^t,$$

where we have put $\alpha=P_t{}^t=g^{tk}P_{tk}$ and $Q_t{}^t=g^{tk}Q_{tk}$.

Transvecting (2.8) with g^{ji}, we obtain $(4n-2)Q_t{}^t=0$, from which,

(2.9) $$Q_t{}^t=0.$$

Substituting (2.9) into (2.8), and taking account of the fact that φ_{ji} is skew-symmetric with respect to j and i, we obtain

$$Q_{ji}+Q_{ij}=0 \tag{2.10}$$

provided that $n>1$.

Substituting (2.9) and (2.10) into (2.8), we obtain

$$Q_{ji}=\frac{\alpha}{2n}\varphi_{ji}, \tag{2.11}$$

from which,

$$P_{ji}=\frac{\alpha}{2n}g_{ji} \tag{2.12}$$

by the help of (0.5).

substituting (2.10), (2.11) and (2.12) into (2.6), we find

$$'K_{kji}{}^{h}=K_{kji}{}^{h}+\frac{\alpha}{2n}(\delta_j^h g_{ki}-\delta_k^h g_{ji}+\varphi_j{}^h\varphi_{ki}-\varphi_k{}^h\varphi_{ji}+2\varphi_i{}^h\varphi_{kj}). \tag{2.13}$$

Contracting (2.13) with respect to h and k, we find

$$'K_{ji}=K_{ji}-\frac{n+1}{n}\alpha g_{ji}. \tag{2.14}$$

Transvecting (2.14) with g^{ji}, we find

$$'K=K-2(n+1)\alpha. \tag{2.15}$$

We now define the Bochner curvature tensor formed with $'\Gamma_{ji}^h$ by

$$B_{kji}{}^{h}={}'K_{kji}{}^{h}+\delta_k^h{}'L_{ji}-\delta_j^h{}'L_{ki}+{}'L_k{}^h g_{ji}-{}'L_j{}^h g_{ki}+\varphi_k{}^h{}'M_{ji}-\varphi_j{}^h{}'M_{ki}+{}'M_k{}^h\varphi_{ji}-{}'M_j{}^h\varphi_{ki}-2({}'M_{kj}\varphi_i{}^h+\varphi_{kj}{}'M_i{}^h), \tag{2.16}$$

where

$$'L_{ji}=-\frac{1}{2(n+2)}\left[{}'K_{ji}+\frac{1}{4(n+1)}{}'Kg_{ji}\right], \tag{2.17}$$

$$'K_{ji}={}'K_{tji}{}^{t},\quad 'K={}'K_{ji}g^{ji},\quad 'L_j{}^h={}'L_{jt}g^{th}, \tag{2.18}$$

and

$$'M_{ji}=-{}'L_{jt}\varphi_i{}^t,\quad 'M_j{}^h={}'M_{jt}g^{th}. \tag{2.19}$$

Substituting (2.14) and (2.15) into (2.17) and taking account of (2.2), we find

$$'L_{ji}=L_{ji}+\frac{\alpha}{4n}g_{ji}, \tag{2.20}$$

from which

$$'M_{ji}=M_{ji}+\frac{\alpha}{4n}\varphi_{ji} \tag{2.21}$$

by virtue of (2.4) and (2.19).

Substituting (2.13), (2.20) and (2.21) into (2.16) and comparing the result with (2.1), we obtain

$$'B_{kji}{}^{h}=B_{kji}{}^{h}.$$

Thus we proved completely the main theorem.

References

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Kook Min University,
Sung Kyun Kwan University