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OPENNESS OF SURJECTIVE CONTINUOUS LINEAR MAPPINGS

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1. Introduction

One of standard theorems in functional analysis is the open mapping theorem (cf. [5]); namely,

any surjective linear mapping from a Frechet space (a metrizable locally convex complete topological vector space) onto another Frechet space is an open mapping.

In general this theorem is no more true if the relevant spaces are not Frechet spaces (cf. [5]).

In this paper, however, we shall show that if a surjective continuous linear mapping is defined on a dual space of a reflexive Frechet space to another dual space of the same type, then it is an open mapping. (Dual space of a Frechet space is not a Frechet space in general.) In particular, it follows that any surjective continuous linear mapping from $\mathcal{E}'(\Omega)$, the space of distributions with compact support in Ω, onto itself is an open mapping.

2. Notations and preliminaries

Throughout this paper E and F will denote Frechet spaces. We denote by E' and F' the dual spaces of E and F with the strong dual topologies.

When M is a closed subspace of E, we denote by $M^{\perp}$, the *annihilator* of M: $M^{\perp}=\{f\in E'|f(x)=0 \text{ for all } x\in M\}$.

Let u be a continuous linear mapping from E into F. We say that u is a *homomorphism* from E into F (or onto $u(E)$) if it is an open mapping from E onto $u(E)$, a subspace of F, equipped with the relative topology induced by F. When u is an injective homomorphism from E onto $u(E)$, we say that u is an *isomorphism* from E into F (or onto $u(E)$).

The transpose of u from F' into E' will be denoted by ${}^t u$.

We shall use the following theorem without proof (cf. [2], [4]).

THEOREM. (Dieudonné–Schwartz)

Let M be a closed subspace of a reflexive locally convex topological vector *space X. Then M' with the strong dual topology is isomorphic to $X'/M^{\perp}$ with the quotient topology derived from the strong dual topology.*

3. Main theorem

THEOREM. *Let E and F be reflexive Frechet spaces and E' and F' be their dual spaces. Let u be a continuous linear mapping from E into F. If ${}^t u$, the transpose of u, is a surjective continuous linear mapping from F' onto E', then ${}^t u$ is an open mapping with respect to the strong dual topologies on F' and E'.*

Proof. Since ${}^t u$ is surjective, it follows that u is injective and u is a homomorphism from E into F with respect to the weak topologies on E and F (cf. [1]).

Note that when E and F are Frechet spaces, u is a homomorphism from E into F with respect to weak topologies if and only if it is a homomorphism with respect to the original topologies (cf. [3]). Thus u is an iso morphism from E with the original topology onto $u(E)$, a subspace of F, with the relative topology induced by the original topology on F. Therefore, $u(E)$ with the relative topology is complete and hence is a Frechet space.

Now let us consider the following diagram:

$$\begin{array}{ccccc} E' & \xleftarrow{q} & (u(E))' & \xleftarrow{p} & F' \\ \updownarrow & & \updownarrow & & \updownarrow \\ E & \xrightarrow{i} & u(E) & \xrightarrow{j} & F \end{array}$$

where $u=j\circ i$, $q={}^t i$ and $p={}^t j$.

Note that we have ${}^t u=q\circ p$ and that q is an isomorphism from $(u(E))'$ onto E'.

Since $u(E)$ is a closed subspace of F and F is reflexive, we have by the Dieudonné-Schwartz theorem, that $F'/(u(E))^{\perp}$ with the quotient topology is isomorphic onto $(u(E))'$ with the strong dual topology under the map

$$f+(u(E))^{\perp} \longrightarrow f|_{u(E)} \text{ for any } f\in F'.$$

Let $r: F'/(u(E))^{\perp} \longrightarrow (u(E))'$ be the isomorphism. Then $p=r\circ\pi$ where π is the canonical quotient map from F' onto $F'/(u(E))^{\perp}$.

Since ${}^t u=q\circ r\circ\pi$ and π, r, and q are all open, it follows that ${}^t u$ is an open mapping. This completes the proof.

When Ω is a nonempty open subset of R^n, $C^{\infty}(\Omega)$, the space of all the infinitely differentiable functions form a reflexive Frechet space in the usual manner (cf. [6]). The dual of $C^\infty(\Omega)$ is the space $\varepsilon'(\Omega)$ of all distributions with compact support in Ω. We note that $\varepsilon'(\Omega)$ with the strong dual topology is not a Frechet space. Applying the previous theorem to $\varepsilon'(\Omega)$, we get the following.

COROLLARY. *Any surjective continuous linear map from* $\varepsilon'(\Omega)$ *onto* $\varepsilon'(\Omega)$ *is an open mapping with respect to strong dual topolgy on* $\varepsilon'(\Omega)$

Proof. Apply the theorem to

$$^t u : C^\infty(\Omega) \to C^\infty(\Omega).$$

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