Bull. Korean Math. Soc.
Vol. 16, No. 1, 1979

ON OPERATOR SEMISTABLE PROBABILITY MEASURES ON HILBERT SPACES

BY SANG GONE LEE

1. Introduction and notation

Throughout H and H^* will denote a real separable Hilbert space and its topological dual, respectively; and $M(H)$ will denote the class of probability (prob.) measures on $\mathfrak{B}(H)$, the smallest σ-algebra containing the open sets of H. For $\mu\in M(H)$, the *characteristic functional* (ch. f.) of μ denoted by $\hat{\mu}$ is a complex valued function on H^* defined by

$$\hat{\mu}(y)=\int_H e^{i(x,y)}d\mu(x)$$

where (,) is the inner product on H and $y\in H^*$. For any μ, $\nu\in M(H)$, we denote by $\mu*\nu$ the *convolution* of μ and ν which is a probability measure on $\mathfrak{B}(H)$ defined by

$$\mu*\nu(B)=\int_H \mu(B-x)\ d\nu(x)$$

for every $B\in\mathfrak{B}(H)$. The symbol μ^{*n} will denote μ convoluted n times with itself. It is well known [2, p. 152] that every $\mu\in M(H)$ is uniquely determined by its ch. f. $\hat{\mu}$, and $\widehat{\mu*\nu}(\cdot)=\hat{\mu}(\cdot)\hat{\nu}(\cdot)$. For μ, $\mu_n\in M(H)$, $n=1, 2, \cdots$, we shall say that μ_n *converges weakly to* μ (in symbols, $\lim_n \mu_n=\mu$) if for every bounded continuous real valued function f on H, $\int f\,d\mu_n \longrightarrow \int f\,d\mu$ as $n\to\infty$. It is well known that $M(H)$ is the topological semigroup with the topology of weak convergence of measures and the convolution of measures as a multiplication.

Let A be a bounded, invertible, linear operator on H. For any $\mu\in M(H)$, we denote by $A\mu$ the measure on $\mathfrak{B}(H)$ defined by $A\mu(B)=\mu(A^{-1}(B))$ for every $B\in\mathfrak{B}(H)$. It is not difficult to verify that for any bounded linear operator A on H, $A(\mu*\nu)=A\mu*A\nu$ and $\widehat{A\mu}(y)=\hat{\mu}(A^*y)$, where A^* denotes the adjoint of A; i. e. $(Ax, y)\equiv(x, A^*y)$. The measure δ_x defined by $\delta_x(B)=0$ if $x\in B$, and 1 if $x\in B$, is said to be the *degenerated probability measure* at $x\in H$.

Received May 4, 1979

Characteristic functions which satisfy, for all t, an equation of the form $f(t)=\{f(bt)\}^a$ where $a>0$ and $0<b<1$ were considered by Lévy [1937], and the solutions have been called semistable laws. Pillai [3] has considered semistable laws and proved that they can be identified as weak limits of laws. Recently, Kumar [1] has extended this result to real separable Hilbert spaces.

Motivated from the work of Kumar [1], Pillai [3] and Lévy, in this paper we define operator r-semistable prob. measures on H and obtain a characterization of these measures in terms of the weak limits of measures in $M(H)$.

2. Preliminaries

In this section we collect necessary definitions and some known results that will be needed in this paper. We begin with the following definitions.

DEFINITION 2.1. A positive semidefinite Hermitian operator A is said to be an S-operator if it has finite trace, i.e., for some orthonormal basis $\{e_j\}$, $\sum_{i=1}^{\infty}(Se_i, e_i)<\infty$.

DEFINITION 2.2. A family $\{S_\alpha\}$ of S-operators is said to be *compact* if the following conditions are satisfied:

(i) $\sup_\alpha \text{trace}(S_\alpha)<\infty$;

(ii) $\lim_{N\to\infty} \sup_\alpha \sum_{i=N}^{\infty}(S_\alpha e_i,e_i)=0$

for some complete orthonormal sequence $\{e_n\}$ in H.

DEFINITION 2.3. A measure $\mu\in M(H)$ is said to be *infinitely divisible* if for each positive integer n, there exists a measure λ_n in $M(H)$ such that $\mu=\lambda_n^{*n}$

Now we will state some known results which will be needed in the proof of our theorems in this paper.

THEOREM 2.4. [2] *A function* $\phi(y)$ *is the characteristic function of an infinitely divisible measure* μ on H *if and only if it is of the form*

$$\phi(y)=\exp\{i(x_o, y)-\frac{1}{2}(Sy, y)+\int k(x, y)\ dM(x)\},$$

where x_o *is a fixed element of* H, S *is an* S*-operater and* M *is a* σ*-finite measure with finite mass outside every neighborhood of the origin and*

$$\int_{\|x\| \leq 1} \|x\|^2 \, dM(x) < \infty$$

Here, $k(x, y)$ *is the function* $\{e^{i(x,y)} - 1 - (i(x,y)/1+\|x\|^2)\}$.
The above representation is unique.

If μ is any infinitely divisible measure on H, by $\mu = [x, S, M]$, we mean that the three quantities occurring in the representation of Theorem 2.4 are, respectively, $x, S,$ *and* M.

REMARK 2.5. If μ is an infinitely divisible measure with the representation $[x_0, S, M]$, it follows from Theorem 2.4 that for any $t>0$, $[tx_0, tS, tM]$ is the ch. f. of some infinitely divisible prob. measure on H. In view of this, we denote by μ^t the infinitely divisible prob. measure with the represenation $[tx_0, tS, tM]$.

REMARK 2.6. If $\mu = [x, S, M]$ and T denotes the associated operator defined by

$$(Ty, y) = (Sy, y) + \int_{\|x\| \leq 1} (x, y)^2 \, dM(x), \quad y \in H,$$

then it is easily shown that T is an S-operator.

THEOREM 2.7. [2] *In order that a sequence of infinitely divisible measure* $\mu_n = [x_n, S_n, M_n]$ *converges weakly to* $\mu = [x_0, S_0, M_0]$ *it is necessary and sufficient that*

(i) $\lim_n x_n = x_0$,

(ii) $M_n \Longrightarrow M_0$ *outside every closed neighborhood* N *of the origin*

(iii) *the sequence of* S-*operators* $\{T_n\}$ *in Remark* 2.6 *is compact*

(iv) $\lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \int_{\|x\| \leq \varepsilon} (x, y)^2 \, dM_n + (S_n y, y) = (S_0 y, y)$

3. A characterization of operator r-semistable probability measures on H

In this section we define operator r-semistable prob. measures on H and obtain a characterization of these measures.

DEFINITION 3.1. Let $\mu \in M(H)$ be non-degenerate infinitely divisible and $0 < r < 1$, then, we say that μ is *operator r-semistable* if there exists a bounded, invertible linear operator B on H such that

$$\mu^r = B\mu \tag{3.1}$$

It is clear from the definition that an operator r-semistable measure is also operator r^n-semistable for every positive integer n.

THEOREM 3.2. *Let $\mu \in M(H)$ be non-degenerate infinitely divisible. Then $\{\mu^t : t \in (0, \infty)\}$ is a semigroup and weakly continuous.*

Proof. It is clear from the definition of μ^t in Remark 2.6 that $\{\mu^t : 0 < t < \infty\}$ is a semigroup. Now we proceed to prove the weak continuity. Let $\mu = [x_0, S, M]$, and let $\{t_n\}$ be a sequence in $(0, \infty)$ converging to t. Then we must show that $\mu^{t_n} = [t_n x_0, t_n S, t_n M]$ weakly converges to $\mu^t = [t x_0, tS, tM]$.

To show this, it is enough to verify the conditions (i) – (iv) in Theorem 2.7. The conditions (i) and (ii) are easy to be verified. Now, verify the condition (iii): by noting the easy facts that $T_n = t_n T$ for $n = 1, 2, \cdots$, and $\sup_n \{t_n\} = u < \infty$, we easily see that

(i) $\sup_n \operatorname{Trace}(T_n) = \sup_n t_n \operatorname{Trace}(T) = u \operatorname{Trace}(T) < \infty$

(ii) $$\lim_{N\to\infty} \sup_n \sum_{i=N}^{\infty} (T_n e_i, e_i) = \lim_{N\to\infty} \sup_n t_n \sum_{i=N}^{\infty} (Te_i, e_i) = u \lim_{N\to\infty} \sum_{i=N}^{\infty} (Te_i, e_i) = 0$$

Hence $\{T_n\}$ is compact. Finally, verify the condition (iv):

$$\lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} \int_{\|x\|<\varepsilon} (x, y)^2 \, d(t_n M) + (t_n S y, y)$$

$$= \lim_{\varepsilon\to 0} \overline{\lim_{n\to\infty}} t_n \left(\int_{\|x\|<\varepsilon} (x, y)^2 \, dM + (Sy, y)\right)$$

$$= t\left((Sy, y) + \lim_{\varepsilon\to 0} \int_{\|x\|<\varepsilon} (x, y)^2 \, dM\right) = t(Sy, y).$$

Now we state and prove our main theorem in this paper.

THEOREM 3.3. *Let $\mu \in M(H)$ be non-degenerate infinitely divisible and $0 < r < 1$. Then the following is equivalent:*

(i) *μ is operator r-semistable.*

(ii) *there exist a bounded, invertible, linear operator B and an increasing sequence $\{k_n\}$ of positive integers such that*

$$\lim_{n\to\infty} \frac{k_{n+1}}{k_n} = r \tag{3.2}$$

and

$$\lim_{n\to\infty} B^n \mu^{*k_n} = \mu \tag{3.3}$$

Proof. (i) implies (ii): By the definition of operator r-semistable prob. measure, there exists a bounded, invertible, linear operator B on H such that $\mu^r = B\mu$. By iterating (3.1) n times $(n=1, 2, \cdots)$, we obtain

$$\mu^{r^n} = B^n \mu. \tag{3.4}$$

Now choose an increasing sequence $\{k_n\}$ of positive integers such that $r^n \cdot k_n \to 1$ as $n \to \infty$ (for example, k_n=the integral part of $\frac{1}{r^n}$). Then clearly, $\{k_n\}$ satisfies (3.2). By taking k_n-th power on both sides of the equation (3.4), and letting $n \to \infty$, we obtain, by Theorem 2.7,

$$\lim_{n \to \infty} B^n \mu^{*k_n} = \mu.$$

(ii) implies (i): Let $\lim_n B^n \mu^{*k_n} = \mu$, for some bounded, invertible, linear operator B on H and $\{k_n\}$ satisfying (3.2). Then each $y \in H^*$

$$\begin{aligned}\hat{\mu}(y) &= \lim_{n \to \infty} \widehat{B^n \mu^{*k_n}}(y) = \lim_{n \to \infty} \{\hat{\mu}(B^{*n} y)\}^{k_n} \\ &= \lim_{n \to \infty} \{\hat{\mu}(B^{*n}(B^* y))\}^{k_n \cdot k_{n+1}/k_n} \\ &= \{\hat{\mu}(B^* y)\}^r\end{aligned}$$

By replacing y by $(B^*)^{-1} y$, we obtain

$$\hat{\mu}((B^*)^{-1} y) = \{\hat{\mu}(y)\}^r.$$

Since every $\mu \in M(H)$ is uniquely determined by its ch. f., it follows that $\mu^r = B^{-1}\mu$. Hence μ is operator r-semistable.

REMARK 3.4. (i) It is worth pointing out that while proving (ii) $\Longrightarrow$ (i) above, we need only assume that r is positive and not necessarily belongs to $(0, 1)$. However, if we assume that the norm of B, $\|B\|$ is less than 1, then $r \in (0, 1)$. This follows from the following argument: Suppose $r \geq 1$; then by (3.4), we have

$$|\hat{\mu}(B^{*n} y)| = |\hat{\mu}(y)|^{r^n} \leq |\hat{\mu}(y)|$$

for all $y \in H^*$. Since $\|B^*\| = \|B\| < 1$, $\|B^{*n}\| \to 0$ as $n \to \infty$ and hence $|\hat{\mu}(y)| = 1$ for all $y \in H^*$. This implies that μ is degenerate, contradicting the non-degeneracy of μ. Hence we must have $r \in (0, 1)$.

(ii) If we take a bounded, invertible, linear operator B on H as T_a: $x \longrightarrow ax$, (a is a real number), we obtain a result of Kumar [1] from our main Theorem. Hence Theorem 3.3 is a generalization of Theorem 2.1 in [1].

References

1. A. Kumar, *Semi-stable probability measures on Hilbert Spaces*, J. Multivariate Anal., **6**(1976), 309-318
2. K. R. Parthasarathy, *Probability Measures in Metric Spaces*, New York, Academic Press, 1967
3. R. N. Pillai, *Semi-stable laws as limit distributions*, Annals of Mathematical Statistics, **42**(1971), 780-783

Jeon Bug National University