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SEPARATION AXIOMS IN BITOPOLOGICAL SPACES

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J. C. Kelly [1] initiated a study of bitopological spaces. A set X equipped with two topologies is called a bitopological space. Separation axioms in bitopological spaces have been studied by various authers, e. g., J. C. Kelly [1], C. W. Patty [3], E. P. Lane [2] and others. Concepts of pairwise Hausdorff, pairwise regular and pairwise normal were introduced by J. C. Kelly; pairwise completely regular and pairwise perfectly normal were defined by E. P. Lane; pairwise complete normality was introduced by C. W. Patty. The purpose of this paper is to introduce the concept of total normality for bitopological spaces and to study some related results. Generally, terms and notation not explained herein are taken from I. L. Reilly [5].

We first generalize the concept of a zero-set.

DEFINITION 1. Let (X, T_1, T_2) be a bitopological space. If f is a real-valued function on X that is T_1-lsc and T_2-usc then $\{x\in X|f(x)\leq 0\}$ is a T_1-zero-set [w. r. t. T_2], and $\{x\in X|f(x)\geq 0\}$ is a T_2-zero-set [w. r. t. T_1]. A subset is called a T_i-cozero-set if its complement is a T_i-zero-set, $i=1, 2$.

It is immediate from the definition that a T_1-zero-set is T_1-closed and a T_2-zero-set is T_2-closed.

Let g be a T_1-lsc and T_2-usc function on X. Then, because

$$\{x\in X|g(x)\leq 0\} = \{x\in X|f(x)=(g^{\vee}0)(x)=0\},$$

any T_1-zero-set is of the form $Z(f)=\{x\in X|f(x)=0\}$, where f is T_1-lsc and T_2-usc and $f\geq 0$, Similarly, any T_2-zero-set is of the form $Z(f)=\{x\in X|f(x)=0\}$, where f is T_1-usc and T_2-lsc and $f\geq 0$.

Another characterization of pairwise normality is given by the following generalization of Urysohn's Lemma.

THEOREM 2. *(X, T_1, T_2) is pairwise normal iff for each T_1-closed set A and T_2-closed set B disjoint from A there exist T_1-zero-set E, T_2-zero-set F, T_2-cozero-set U and T_1-cozero-set V such that*

$$A\subset E\subset U,\quad B\subset F\subset V \text{ and } U\cap V=\phi.$$

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The following result can sometimes be used to extend results on pairwise normal spaces to more general bitopological spaces.

THEOREM 3. *Let (X, T_1, T_2) be a bitopological space. If A is a T_1-zero-set that is disjoint from the T_2-zero-set B, then there exist a T_2-cozero-set U and a T_1-cozero-set V disjoint from U such that*

$$A \subset U \text{ and } B \subset V.$$

Proof. Let $A=Z(f)$, where f is T_1-lsc and T_2-usc and $f \geq 0$. Similarly, let $B=Z(g)$, where g is T_1-usc and T_2-lsc and $g \geq 0$. Since $A \cap B=\phi$, it follows that $f(x)+g(x)>0$ for all $x \in X$. Thus we can define a function $h: X \to [0,1]$ by

$$h(x)=f(x)/(f(x)+g(x)), \quad \text{if } x \in X.$$

Then $h=0$ on A and $h=1$ on B. And if $x \in B$ then $h(x)=1$, but because $h \leq 1$ it follows that h is T_2-usc at $x \in B$. If $g(x) \neq 0$ then

$$h(x)=1-\frac{1}{(f(x)/g(x))+1}.$$

Because $g>0$ on $X-B$, it follows that $1/g$ is T_1-lsc and T_2-usc on $X-B$. Thus f/g is T_1-lsc and T_2-usc on X-B, and consequently $1/((f/g)+1)$ is T_1-usc and T_2-lsc on X-B. This proves that $h|(X-B)$ is T_1-lsc and T_2-usc.

If $x \in X-B$ and $\varepsilon>0$, there is a subset G of $X-B$ such that G contains x, G is T_2-open in $X-B$, and if $y \in G$ then $h(y)<h(x)+\varepsilon$. But G is T_2-open in X, so h is T_2-usc at $x \in X-B$. Therefore h is T_2-usc. Similarly, if $x \in A$ then $h(x)=0$ and $h \geq 0$, so h is T_1-lsc at x, If $x \in X-A$, the fact that $h(X-A)=1/((g/f)+1)$ shows that h is T_1-lsc at x. Thus h is T_1-lsc.

Let $U=\{x \in X \mid h(x)<1/2\}$ and $V=\{x \in X \mid h(x)>1/2\}$.

Then U and V are appropriately cozero, disjoint and separate A and B.

We say that a subset A of (X, T_1, T_2) is a T_i-G_δ-set if A is a countable intersection of T_i-open sets. Dually, a subset B is a T_i-F_σ-set if B is a countable union of T_i-closed sets, $i,j=1,2$.

THEOREM 4. *A T_i-zero-set of (X, T_1, T_2) is a T_i-closed T_j-G_δ-set, $i,j=1,2$, $i \neq j$.* And a *T_i-closed T_j-G_δ-set in a pairwise normal space is a T_i-zero-set, $i,j=1,2$, $i \neq j$.*

Proof. Let A be a T_i-zero-set in a bitopological space X. Then A is evidently T_i-closed and if $A=Z(g)$, where g is T_i-lsc and T_j-usc on X

such that $g\geq 0$, then $A=\bigcap_{n\in N} G_n$, where $G_n=\{x\in X|g(x)<1/n\}$, so that A is a T_j-G_δ-set.

Now let A be a T_i-closed set of a pairwise normal space X such that $A=\bigcap_{n\in N} G_n$, where each G_n is a T_j-open set. For each n there exist a T_1-lsc and T_2-usc function f_n on X such that $f_n(A)=0$, $f_n(G_n^C)=1/2^n$ and $f_n(X)\subset[0, 1/2^n]$. Then the function $f:X\to[0,1]$ defined by $f(x)=\sum_{n=1}^{\infty} f_n(x)$ is T_1-lsc and T_2-usc, and it is clear that $A=Z(f)$.

If A is a T_i-clsed set in a pairwise normal space (X, T_1, T_2) and U is a T_i-open set such that $A\subset U$, then by Theorem 2 and Theorem 4, there exists a T_j-open T_i-F_σ-set H such that $A\subset H\subset U$, where $i, j=1, 2$, $i\neq j$. We consider generalization of this situation.

DEFINITION 5. A subset M of a bitopolgical space (X, T_1, T_2) is a *generalized T_i-F_σ-set* if for each T_j-open set U such that $M\subset U$ there exists a T_i-F_σ-set F such $M\subset F\subset U$.

DEFINITION 6. A subset M of a bitopological space (X, T_1, T_2) is (i, j) *normally situated* in X if for each T_j-open set U such that $M\subset U$ there exists a T_j-open set G such that $M\subset G\subset U$ and $G=\subset_{\lambda\in\Lambda} G_\lambda$, where $\{G_\lambda|\lambda\in\Lambda\}$ is a family, locally finite in G, of T_j-open T_i-F_σ-sets of X, $i, j=1, 2$, $i\neq j$. A subset M of X is *normally situated* in X if M is $(1, 2)$ normally situated in X and $(2, 1)$ normally situated in X.

THEOREM 7. *If M is a generalized T_i-F_σ-set in a pairwise normal space X, then M is (i, j) normally situated in X, $i, j=1, 2$, $i\neq j$.*

Poof. If M is a generalized T_i-F_σ-set and $M\subset U$, where U is a T_j-open set, then there exists a family $\{A_n|n\in N\}$ of T_i-closed set such that $M\subset\bigcup_{n\in N} A_n\subset U$. But since X is a pairwise normal space, $A_i\subset H_i\subset U$ for each i, where H_i is a T_j-open T_i-F_σ and $M\subset\bigcup_{i\in N} H_i\subset U$. Hence M is a (i, j) normally situated in X, $i, j=1, 2$, $i\neq j$.

DEFINITION 8. A pairwise normal space (X, T_1, T_2) is said to be *pairwise totally normal* if every subset of X is normally situated in X.

Equivalently, a pairwise normal space X is pairwise totally normal if each T_j-open set G is the union of a family $\{G_\lambda|\lambda\in\Lambda\}$, locally finite in G, of T_j-open T_i-F_σ-sets of X.

THEOREM 9. (X, T_1, T_2) *is pairwise regular if it is pairwise totally normal.*

Proof. If G is a T_i-open set and $x \in G$, then $x \in U \subset G$, where U is a T_i-open $T_j - F_\sigma$-set. Hence there exists a T_j-closed set F such that $x \in F \subset G$. Since X is pairwise normal there exists a T_j-open set V such that $F \subset V \subset \bar{V}^i \subset G$. Then $x \in V \subset \bar{V}^i \subset G$ and hence X is pairwise regular.

If X is a pairwise totally normal space then every subspace of X is pairwise totally normal. In fact, suppose A is a subspace of X and suppose that $M \subset U \subset A$, where U is T_j-open in A. Then there exists V a T_j-open in X such that $U = V \cap A$. Since M is normally situated in X and $M \subset V$, there exists a family $\{G_\lambda \mid \lambda \in \Lambda\}$, locally finite in $G = \bigcup_{\lambda \in \Lambda} G_\lambda$, of T_j-open $T_i - F_\sigma$-sets of X such that $M \subset G \subset V$. Then $G \cap A = \bigcup_{\lambda \in \Lambda} (G_\lambda \cap A)$ and for each λ the set $G_\lambda \cap A$ is a T_j-open T_i-F_σ-set of A. Since $M \subset G \cap A \subset U$ it follows that M is normally situated in A. Hence A is pairwise totally normal.

THEOREM 10. (X, T_1, T_2) *is pairwise totally normal if it is pairwise perfectly normal.*

Proof. Let M be a subset of X and $M \subset U$, where U is T_j-open set. Since X is pairwise perfectly normal, U is T_j-open $T_i - F_\sigma$-set. Thus M is a generalized $T_i - F_\sigma$-set so that M is normally situated in X by Theorem 7.

Hence X is a pairwise totally normal space.

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