Bull. Korean Math. Soc.
Vol. 16, No. 1, 1979

ON GENERALIZED HEISENBERG GROUPS

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1. Introuction

Let G be a finite group. Let K be the complex number field and K^* its multiplicative group. A mapping $\alpha: G\times G \to K^*$ is called a 2-cocycle on G if

$$\alpha(g, h)\,\alpha(gh, k) = \alpha(g, hk)\,\alpha(h, k)$$

for all g, h, k in G. Given a 2-cocycle α on G we let $K^\alpha[G]$ denote the twisted group algebra of G over K with respect to α. That is, $K^\alpha[G]$ is a K-algebra with K-basis $\{\bar{g} \mid g\in G\}$ and with multiplication defined distributively and using

$$\bar{g}\bar{h} = \alpha(g, h)\,\overline{gh}$$

for all g, h in G. The definition of a 2-cocycle is that which makes the algebra $K^\alpha[G]$ associative. In particular, if $\alpha(g, h)=1$ for all g, h in G then $K^\alpha[G]$ is in fact $K[G]$, the ordinary group algebra of G over K.

It is easy to show that the twisted group algbra $K^\alpha[G]$ is semisimple. Hence

$$K^\alpha[G] \cong \mathrm{Mat}_{n_1}(K)\oplus\cdots\oplus\mathrm{Mat}_{n_r}(K)$$

for some positive integers $n_1, \cdots, n_r$, and the center $Z(K^\alpha[G])$ of $K^\alpha[G]$ is r dimensional over K.

The center of the ordinary group algebra $K[G]$ has a K-basis consisting of the class sums. Hence the center of $K[G]$ is one dimensional over K if and only if G is a trivial group. It is more difficult to find the center of the twisted group algebra $K^\alpha[G]$. We call group G a *generalized Heisenberg group* if there is a 2-cocycle α on G such that the center of $K^\alpha[G]$ is one dimensional over K. Note that G is a generalized Heisenberg group if and only if there is a 2-cocycle α on G such that $K^\alpha[G]\cong \mathrm{Mat}_m(K)$ and $|G| = m^2$ for some positive integer m.

*) This research is supported by KOSEF Research Grant.

The generalized Heisenberg groups are closely related to the groups of central type. That is the reason why the character theory of finite groups is used to study the generalized Heisenberg groups. It has been conjectured in [11] that any group of central type must be solvable.

In this paper we will discuss the structure of the center of $K^\alpha[G]$. We also study the relation between the generalized Heisenberg groups and the groups of central type. Several examples of generalized Heisenberg groups are considered.

In the last section of this paper, we will prove a theorem on projective lattices over the group ring ZG where G is the generalized quaternion group of order 16 and Z is the ring of integers.

The notation in this paper is standard. The group G is assumed to be finite. The order of G is denoted by $|G|$. Let g and h be elements of G. We define $g^h=h^{-1}gh$ and $[g,h]=g^{-1}h^{-1}gh$. The centralizer of g in G is denoted by $C_G(g)$. Thus $C_G(g)=\{x\in G\,|\,gx=xg\}$. The commutator subgroup of G is denoted by G'.

2. The center of the twisted group algbra

Let G be a finite group. Let K be the complex numbe field and K^* its multiplcative group. Let $K^\alpha[G]$ denote the twisted group algebra of G over K with respect to a 2-cocycle α on G. Then $U=\{a\,\bar{g}\,|\,a\in K^*,\ g\in G\}$ is a multiplicative subgroup of the group of units of $K^\alpha[G]$. Moreover, the mapping $\pi: U\to G$ defined by $\pi(a\bar{g})=g$ is a homomorphism of U is a central extension of G with kernel W, where W is isomorphic to K^*.

Let $a\,\bar{g}$ be any element of U. Then

$$\pi(C_U(a\bar{g}))=\{x\in G\,|\,\bar{g}\bar{x}=\bar{x}\bar{g}\}.$$

An element $g\in G$ is said to be *α-special* if $\alpha(g,x)=\alpha(x,g)$ for every $x\in C_G(g)$. It is easy to see that an element $g\in G$ is α-special if and only if $C_G(g)=\pi(C_U(a\bar{g}))$ for any $a\in K^*$. And it is clear that if g is α-special then so is every conjugate of g in G.

LEMMA 1. *Let $\mathscr{C}$ be a conjugacy class of G. Then $\mathscr{C}$ consists of α-special elements if there exists a function $\pi:\mathscr{C}\to K^*$ such that*

$$\lambda(g)\,\alpha(g,h)=\lambda(g^h)\,\alpha(h,g^h)$$

for all $g\in\mathscr{C}$ and $h\in G$.

Proof. Suppose that $\mathscr{C}$ consists of α-special elements. Then for all $g\in\mathscr{C}$ and $a\in K^*$, we have $C_G(g)=\pi(C_U(a\bar{g}))$ and

$$|G : C_G(g)| = |U : C_U(a\bar{g}))|.$$

Therefore, if $\mathscr{K}$ is the conjugacy class in U containing $a\bar{g}$ then $|\mathscr{K}| = |\mathscr{C}|$ and $\pi(\mathscr{K}) = \mathscr{C} = \{g^h \mid h \in G\}$. This implies that

$$\mathscr{K} = \{\lambda(g^h)\, g^h \mid h \in G\}$$

for some function $\lambda : \mathscr{C} \to K^*$. Since $\bar{h}^{-1}\lambda(g)\bar{g}\bar{h}$ is contained in $\mathscr{K}$ and it is a multiple of $\overline{g^h}$, we have

$$\bar{h}^{-1}\lambda(g)\bar{g}\bar{h} = \lambda(g^h)\overline{g^h}.$$

This yields that $\lambda(g)\ \alpha(g, h)\ \overline{gh} = \lambda(g^h)\ \alpha(h, g^h)\ \overline{g^h}$. Hence

$$\lambda(g)\ \alpha(g, h) = \lambda(g^h)\ \alpha(h, g^h)$$

for all $h \in \mathscr{C}$ and $g \in G$.

Conversely, now assume that there exists a function $\lambda : \mathscr{C} \to K^*$ such that $\lambda(g)\ \alpha(g, h) = \lambda(g^h)\ \alpha(h, g^h)$ for all $g \in \mathscr{C}$ and $h \in G$. Let $g \in \mathscr{C}$. Then for all $x \in C_G(g)$ we have $\lambda(g)\ \alpha(g, x) = \lambda(g)\ \alpha(x, g)$ and $\alpha(g, x) = \alpha(x, g)$.

Hence g is α-special, and the class $\mathscr{C}$ consits of α-special elements.

THEOREM 1. *The dimension of the center $Z(K^\alpha[G])$ of the twisted group algebra $K^\alpha[G]$ is equal to the number of conjugacy classes of α-special elements in G.*

Proof. Let $\mathscr{C}_1, \cdots, \mathscr{C}_r$ be the conjugacy classes of α-special elements in G. By Lemma 1, for each $\mathscr{C}_i$ there is a function $\lambda_i : \mathscr{C}_i \to K^*$ such that

$$\lambda_i(g)\ \alpha(g, h) = \lambda_i(g^h)\ \alpha(h, g^h)$$

for all $g \in \mathscr{C}_i$ and $h \in G$. Let $C_i = \sum_{g \in \mathscr{C}_i} \lambda_i(g)\bar{g} \in K^\alpha[G]$. Then the C_i lie in $Z(K^\alpha[G])$ since

$$\bar{h}^{-1}\lambda_i(g)\bar{g}\bar{h} = \lambda_i(g^h)\overline{g^h}$$

for all $g \in \mathscr{C}_i$ and $h \in G$. Moreover, the C_i are linearly independent.

If $z = \sum a_g \bar{g}$, $a_g \neq 0$, is an element of $Z(K^\alpha[G])$, then for all $h \in G$ we have $z\bar{h} = \bar{h}z$ and

$$\sum a_g\, \alpha(g, h)\overline{gh} = \sum a_g\, \alpha(h, g)\overline{hg}.$$

Hence $\alpha(g, x) = \alpha(x, g)$ for all $x \in C_G(g)$, which implies that g is α-specal and $g \in \mathscr{C}_i$ for some i. Since $\bar{h}^{-1}z\bar{h} = z$, it follows that

$$\lambda_i(g)\, a_{g^h} = a_g \lambda_i(g^h)$$

for all $g \in \mathscr{C}_i$ and $h \in G$. Hence $z = \sum a\bar{g} = \sum a_i C_i$ for some $a_i \in K^*$ and thus the C_i span $Z(K^\alpha[G])$. Therefore, the C_i form a basis for $Z(K^\alpha[G])$.

From Theorem 1, it follows that a group G is a generalized Heisenberg group if there is a 2-cocycle α on G such that the only α-special element of G is the identity.

3. The generalized Heisenberg groups

Let be a finite group with center $Z(G)$ and K the complex number field. Let $\mathrm{Irr}(G)$ denote the set of all irreducible complex characters of G. For each $\chi\in\mathrm{Irr}(G)$ let $Z(\chi)=\{g\in G|\ |\chi(g)|=\chi(1)\}$. Then $Z(\chi)$ is a normal subgroup of G containing $\ker\chi=\{g\in G|\chi(g)=\chi(1)\}$.

LEMMA 2. *Let* $\chi\in\mathrm{Irr}(G)$. *Then*

(1) $Z(\chi)/\ker\chi=Z(G/\ker\chi)$ *and it is cyclic.*

(2) $\chi(1)^2\leq|G:Z(\chi)|$.

Equality holds if and only if χ *vanishes on* $G-Z(\chi)$.

(3) *If* $G/Z(\chi)$ *is abelian, then* $\chi(1)^2=|G:Z(\chi)|$.

Proof. It is easy to prove this lemma (see [8, pp. 27-28]).

From Lemma 2 it follows that $Z(G)\subseteq Z(\chi)$ and $\chi(1)^2\leq|G:Z(G)|$. Equality can occur here, and when it does, $Z(G)=Z(\chi)$ and χ vanishes on $G-Z(G)$. We call a group G is of *central type* if there is a character $\chi\in\mathrm{Irr}(G)$ such that $\chi(1)^2=|G:Z(G)|$.

The following therem shows the relation between the generalized Heisenberg groups and the groups of central type.

THEOREM 2. *If* G *is a group of central type, then* $G/Z(G)$ *is a generalized Heisenberg group.*

Conversely, if H *is a generalizd Heisenberg group then there is a group* G *of central type such that* $G/Z(G)\cong H$.

Proof. Assume that G is a group of central type. Then there is $\chi\in\mathrm{Irr}(G)$ such that $\chi(1)^2=|G:Z(G)|=|G/Z(G)|$. As in Lemma 25.4 of [5] we can associate with the ordinary representation $T:G\to GL(V)$ which affords χ a projective representation $T:G/Z(G)\to GL(V)$ with some 2-cocycle α on $G/Z(G)$. Then $T(g)T(h)=\alpha(g,h)T(gh)$ for all g,h in $G/Z(G)$. Let $\bar{g}=T(g)$ for all g in $G/Z(G)$. By Burnside's lemma [5, Theorem 10.1], the set $\{\bar{g}|g\in G/Z(G)\}$ spans the K-vector space $\mathrm{End}_K(V)$ and

$$\dim_K\mathrm{End}_K(V)=(\dim_K V)^2=\chi(1)^2=|G/Z(G)|.$$

Hence $K^\alpha[G/Z(G)]\cong\mathrm{End}_K(V)$ which has center K. Therefore, $G/Z(G)$ is a generalized Heisenberg group.

Conversely, assume that H is a generalized Heisenberg group and let α be a 2-cocycle on H such that $K^{\alpha}[H]$ has center K. Let G be a representation group of H [5, Theorem 25.5]. That is, G is a central extension of H with Kernal M such that every projective representation of H is equivalent to one lifted to G, where M is the Schur multiplier $H^2(H, K^*)$ of H. Since $K^{\alpha}[H]$ has center K, it follows from Theorem 1 that the only α-special element in H is the identity. Therefore, we have $Z(G)=M$ and $G/Z(G)\cong H$. Now let L be a minimal left ideal of $K[H]$. Then the projective representation of H on L lifts to an ordinary representation T of G. This representation T affords a character $\chi\in \mathrm{Irr}(G)$ such that

$$\chi(1)^2=(\dim_K L)^2=\dim_K K[H]=|G:Z(G)|.$$

Thus G is of central type.

The following theorem indicates the importance of determining the generalized Heisenberg p-groups.

THEOREM 3. *The group G is of central type if and only if for each prime p a Sylow p-subgroup S_p of G is of central type and $Z(S_p)=Z(G)\cap S_p$.*

The group H is a generalized Heisenberg group if and only if any Sylow subgroup of H is a generalized Heisenberg group.

Proof. This follows from Theorem 2 and Corollary 4 in [4].

THEOREM 4. *Let G be a finite group such that $G'\subseteq Z(G)$ and $|G'|=p$, where p is a prime. Then $\chi(1)^2=|G:Z(G)|$ for every nonlinear $\chi\in \mathrm{Irr}(G)$.*

In particular, G is of central type and $G/Z(G)$ is a generalized Heisenberg group.

Proof. Let $\chi\in \mathrm{Irr}(G)$ be nonlinear. Then $\ker\chi$ does not contain G'. By assumption, this implies that $\ker\chi\cap G'=1$. Since $Z(\chi)/\ker\chi = Z(G/\ker\chi)$, for all $g\in Z(\chi)$ and $h\in G$ we have

$$[g,h]=g^{-1}h^{-1}gh \in \ker\chi\cap G'=1.$$

This implies $Z(\chi)\subseteq Z(G)$, and hence $Z(\chi)=Z(G)$. Moreover, $G/Z(G)$ is abelian because $G'\subseteq Z(G)$. Therefore, by Lemma 2, we have

$$\chi(1)^2=|G:Z(\chi)|=|G:Z(G)|.$$

A finite p-group G is said to be *extra-special* if $G'=Z(G)$, $|G'|=p$, and G/G' is elementary abelian. For each prime number p, there are two nonisomorphic nonabelian groups of order p^3, both extra-special. Furthermore, every extra-special p-group is the central product of nonabelian p-groups of order p^3, and so has order p^{2m+1} for some positive integer m.

COROLLARY. *Let E be an extra-special p-group of order p^{2m+1} and let be Z a cyclic p-group of order p^k. Let G be the central product of E and Z which is not a direct product. Then*

$$\chi(1)^2 = |G:Z(G)| = p^{2m}$$

for every nonlinear $\chi \in \mathrm{Irr}(G)$. In particular, G is of central type.

Proof. It is clear that $G' = E'$ is of order p and $Z(G) = Z \supseteq G'$. Hence the assertion follows from Theorem 4.

In the above corollary, if Z is of order p then $G = E$. Thus any extra-special p-group is of central type.

4. Projective lattices over the group ring ZG, where G is the generalized quaternion group of order 16

Let $f: R \to R'$ be a ring-homomorphism and M a left R-module. Let $f_{\#}M$ denote the induced left R-module $R' \otimes_R M$. Then there is a canonical R-homomorphism $f_*: M \to f_{\#}M$ given by $f_*(m) = 1 \otimes m$. Let

$$\begin{array}{ccc} R & \xrightarrow{h_1} & R_1 \\ \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle f_1} \\ R_2 & \xrightarrow{f_2} & R' \end{array}$$

be a fiber product of ring-homorphisms, where f_1 or f_2 is surjective. We will give the Milnor's construction [12] of projective modules over R, using projective modules over R_1 and R_2 as building blocks. Given any projective module P_i over R_i $(i=1,2)$ and given an R'-isomorphism $h: f_{1\#}P_1 \to f_{2\#}P_2$, let $P = M(P_1, h, P_2)$ denote the subgroup

$$\{(x_1, x_2) \in P_1 \times P_2 \mid h(f_{1*}(x_1)) = f_{2*}(x_2)\}$$

of the additive group $P_1 \times P_2$. We make P into a left R-module by setting

$$r(x_1, x_2) = (h_1(r)x_1, h_2(r)x_2).$$

Then the following hold:

i) *The module $P = M(P_1, h, P_2)$ is projective over R, Furthermore, if P_1 and P_2 are finitely generated over R_1 and R_2, respectively, then P is finitely generated over R.*

ii) *Every projective R-module is isomorphic to $M(P_1, h, P_2)$ for some P_1, P_2 and h.*

iii) *The module* P_1 *and* P_2 *are isomorphic to* $f_{1*}P$ *and* $f_{2*}P$, *respectively, where* $P=M(P_1, h, P_2)$.

Let $u\in R'^*$, where R'^* is the group of multiplicative units of R'. We denote by u the induced R'-isomorphism

$$\begin{array}{ccc} f_{1*}R_1 & \longrightarrow & f_{2*}R_2 \\ \wr\wr & & \wr\wr \\ R' & \longrightarrow & R' \end{array}$$

given by $x\to xu$ for all $x\in R'$

Now let R be a Dedekind domain whose quotient field K is an algebraic number field. By an *R-order* Λ in a finite dimensional semisimple K-algebra A, we mean a subring of A which is finitely generated R-module and contains a K-basis of a K-vector space A. By a *Λ-lattice* we mean a finitely generated left Λ-module which is torsion-free as an R-module. A Λ-lattice P is called a *locally free* Λ-lattice of *rank* n if for each maximal ideal $\mathfrak{P}$ of R, $P_{\mathfrak{P}}=R_{\mathfrak{P}}\otimes_R P$ is a free $\Lambda_{\mathfrak{P}}(=R_{\mathfrak{P}}\otimes_R\Lambda)$-module with n generators. Clearly, $P_{\mathfrak{P}}$ is an order of $\Lambda_{\mathfrak{P}}$ in A.

If P is locally free, then P is projective. For P is projective if and only if $\mathrm{Ext}^1_\Lambda(P, N)=0$ for all Λ-module N, and the functor commutes with the localization in this setting.

Swan[15] showed that if $\Lambda=RG$ is the group ring of a finite group G over a Dedekind domain R, then every projective Λ-lattice is locally free. Thus the map $rk:\mathrm{K}_0(\Lambda)\to \mathrm{Z}$ defined by

$$\mathrm{rk}([P])=\text{the rank of } P$$

is obviously a well-defined additive epimorphism. The kernel $Cl(\Lambda)$ of this epimorphism is called the *class group* of Λ, or more precisely the *locally free class group* of Λ. The group $Cl(\Lambda)$ is indeed the reduced projtive class group $P(\Lambda)$ of Λ which is defined by Rim in [14].

Swan [15] has shown that if P is a locally free Λ-module, then

$$P=(\text{a free } \Lambda\text{-module})\oplus(\text{a locally free left ideal of } \Lambda).$$

It follows from this fact that every element of $Cl(\Lambda)$ can be written in the form of $[J]-[\Lambda]$ for some locally free left ideal J of Λ. Hence by the Jordan-Zassenhaus theorem [16] the class group $Cl(\Lambda)$ is finite.

Let

$$H_n=\langle x, y\,|\,x^{2^n}=y^2,\ y^4=1,\ yxy^{-1}=x^{-1}\rangle$$

be the generalized quaternion group of order 2^{n+2}, where $n\geq 1$.

Let

$$D_n=\langle x, y\ x^{2^{n+1}}=y^2=1,\ yxy^{-1}=x^{-1}\rangle$$

be the dihedral group of order 2^{n+2}, where $n\geq 0$. Then we have

$$ZH_n/(y^2-1)\cong ZD_{n-1}$$

and

$$ZH_n/(y^2)\cong \mathrm{F}_2D_{n-1},$$

where F_2 is the finite field with two elements. Furthermore, the diagram of canonical ring-homomorphisms

$$\begin{tikzcd} ZH_n \arrow[r] \arrow[d] & ZD_{n-1} \arrow[d] \\ ZH_n/(y^2+1) \arrow[r] & \mathrm{F}_2D_{n-1} \end{tikzcd}$$

is a fiber product, where $\Lambda=ZH_n/(y^2+1)$ is an order in a totally definite quaternion algebra over the field $K=\mathrm{Q}(\eta+\eta^{-1})$, η the 2^n-th primitive root of 1, and its center is $R=\mathrm{Z}[\eta+\eta^{-1}]$. For the details we refer to Fröhlich [6] and Cho [2].

In the rest of this section we will consider the case when $G=H_2$, the generalized quaternion group of order 16. We have $K=\mathrm{Q}(\sqrt{2})$, $R=\mathrm{Z}[\sqrt{2}]$ and $\Lambda=ZG/(y^2+1)$ is the quaternion algebra over $\mathrm{Z}[\sqrt{2}]$. If we denote $x+(y^2+1)$ and $y+(y^2+1)$ in Λ by i and j respectively, then we have $\Lambda=\mathrm{Z}[\sqrt{2}][i,j]$ and it is a $\mathrm{Z}[\sqrt{2}]$-order in the quaternion algebra $A=\mathrm{Q}(\sqrt{2})[1, i, j, k]$.

Krimse [9] has shown that

$$\Sigma=\mathrm{Z}[\sqrt{2}]\left[\frac{1}{\sqrt{2}}(1+i),\ \frac{1}{2}(1+j),\ \frac{1}{2}(1+i+j+k)\right]$$

is a maximal $\mathrm{Z}[\sqrt{2}]$-order in the quaternion algebra A. Furthermore, he has shown that every left ideal in Σ is principal and that Σ contains Λ.

Martinet [10] has shown that every free ZD_1-lattice is free and Cho [2] has proved that every projective Λ-lattice is free. Therefore, every projective lattice of rank 1 are ideals in both Λ and Σ.

Now it is obvious that for every projective ZG-lattice P, there exists a unit $u\in\mathrm{F}_2D_1{}^*$ such that $P\cong M(\Lambda, u, ZD_1)$. Since the localization preserves a fiber product of ring-homomorphisms, every projective ZG-lattice of the form $M(\Lambda, s, ZD_1)$, $s\in\mathrm{F}_2D_1{}^*$, is of rank 1.

Let $j_1:\Lambda\longrightarrow\mathrm{F}_2D_1$ and $j_2:ZD_1\longrightarrow\mathrm{F}_2D_1$ be canonical ring-homomorphisms.

Then $j_1(\Lambda^*)$ and $j_2(ZD_1^*)$ are disjoint. This implies that

$$M(\Lambda,\ s,\ ZD_1)\not\cong M(\Lambda,\ 1,\ ZD_1)\cong ZG,$$

where $s=1+j_1(i+j)$. Hence there are exactly two nonisomorphic ZG–lattices of rank one. But, $|Cl(ZG)|\geq 2$ by [6]. Therefore,

$$Cl(ZG)\cong Z/2Z \text{ and } K_0(ZG)\cong Z\oplus Z/2Z,$$

where $G=H_2$ is the generalized quaternion group of order 16.

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