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TOPOLOGIES ON PRIMITIVE RINGS WITH NONZERO SOCLES

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1. Introduction

Let $\mathfrak{A}$ be a primitive ring with nonzero socle $\mathfrak{S}$. Then $\mathfrak{A}$ can be identified by a dense subring of the ring of linear transformations of a vector space $\mathfrak{M}$ over a division ring Δ, containing the set $\mathcal{E}$ of linear transformations of finite rank belonging to $\mathfrak{A}$.

Let $\mathfrak{M}^*$ be the algebraic conjugate space of $\mathfrak{M}$ and $\mathcal{E}^*$ the set of adjoints F^* in $\mathfrak{M}^*$ of elements F in $\mathcal{E}$. Set $\mathfrak{M}'=\mathfrak{M}^*\mathcal{E}^*$. Then $\mathfrak{M}'$ is total in $\mathfrak{M}^*$. Moreover $\mathfrak{A}$ is contained in the ring $\mathcal{L}_{\mathfrak{M}'}(\mathfrak{M})$ of continuous linear transformations of $\mathfrak{M}$ over Δ, topologized by the $\mathfrak{M}'$-topology.

Let $\mathcal{F}(\mathfrak{M})$ denote the set of all linear transformations of $\mathfrak{M}$ over Δ of finite ranks. It is clear that $\mathcal{F}(\mathfrak{M})$ is an ideal in the ring of linear transformations of $\mathfrak{M}$ over Δ. $\mathcal{F}_{\mathfrak{M}'}(\mathfrak{M})$, defined to be $\mathcal{F}(\mathfrak{M})\cap\mathcal{L}_{\mathfrak{M}'}(\mathfrak{M})$, is an ideal in $\mathcal{L}_{\mathfrak{M}'}(\mathfrak{M})$. The socle $\mathfrak{S}$ of $\mathfrak{A}$ is precisely $\mathcal{F}_{\mathfrak{M}'}(\mathfrak{M})$.

The dual spaces $(\mathfrak{M},\mathfrak{M}')$ is in a sense uniquely determined by the primitive ring $\mathfrak{A}$ with nonzero socle $\mathfrak{S}$ [1].

The purpose of this note is to investigate the relationship among various topologies, i.e., the finite topology in $\mathfrak{A}$, the uniform topology in $\mathfrak{A}$, the dual topology in $\mathfrak{A}$, the uniform topology in $\mathfrak{A}'$, the set of adjoints of elements in $\mathfrak{A}$.

2. Finite topology

Now consider the set $\mathcal{L}(\mathfrak{M},\mathfrak{M}')$. If $x_1, x_2, \cdots, x_m$ and $y_1, y_2, \cdots, y_m$ are finite subset of $\mathfrak{M}$, then we define $0(x_i;y_i)$ to be the set of linear transformations A of $\mathfrak{M}$ into $\mathfrak{M}'$ such that

$$x_iA=y_i,\ i=1,2,\cdots,m.$$

It is clear that intersection of any two $0(x_i,y_i)$ is another one. Then the collection $0(x_i,y_i)$ forms a basis for a topology in $\mathcal{L}(\mathfrak{M},\mathfrak{M}')$. We shall call this topology the finitε topology of $\mathcal{L}(\mathfrak{M},\mathfrak{M}')$. We now note that any open set $0(x_i;y_i)$ is either vacuous or it coincide with an open set $0(x_j;y_j)$ where x_j are linearly independent. For, suppose $x_1, x_2, \cdots, x_r$ is a maximall inearly independent subset of $x_1, x_2, \cdots, x_m$ and let $x=\sum_{j=1}^{r}\beta_{kj}x_j$ hold for $k=r+1, \ldots, m$. Then, unless $y_k=\sum_{j=1}^{r}\beta_{kj}y_j$, $0(x_i;y_i)$ is vacuous. And if the condition hold, then $0(x_i;y_i)=0(x_j;y_j)$, $j=1, \cdots, r$.

This remarks shows that the sets $0(x_j;y_j)$, x_j linearly independent, constitute a basis for our topology. Consider the induced topology on the ring $\mathfrak{A}$. And write $0(x_j;y_j)$ for $0(x_j;y_j)\cap\mathfrak{A}$.

THEOREM 1. *A base of neighborhood of* 0 *of* $\mathfrak{A}$ *in the finite topology are the right ideals* $\{(0;\mathfrak{U})\}$, *where* $\mathfrak{U}$ *is a finite dimensional vector subspace of* $\mathfrak{M}$ *over* Δ.

Proof A base of neighborhood of 0 of $\mathfrak{A}$ are $0(x_j;0)$, x_j linearly independent. Let $\mathfrak{U}$ be a vector subspace of $\mathfrak{M}$ over Δ generated by linearly independent x_j. Then evidently $0(x_j;0)=(0;\mathfrak{U})$.

DEFINITION. Let $\mathfrak{A}$ be a ring. If S is a subset of $\mathfrak{A}$, then

$$\mathfrak{Z}_L(S)=\{z\,|\,z\in\mathfrak{A},\ zs=0 \text{ for all } s\in S\}$$

is called the **left annihilator** of S in $\mathfrak{A}$. And

$$\mathfrak{Z}_R(S)=\{z\,|\,z\in\mathfrak{A},\ sz=0 \text{ for all } s\in S\}$$

is called the **right annihilator** of S in $\mathfrak{A}$. $\mathfrak{Z}_L$ is a left ideal, while $\mathfrak{Z}_R$ is a right ideal in $\mathfrak{A}$.

THEOREM 2. *The base mentioned in Theorem* 1 *coincides with* $\mathfrak{Z}_R(f)$, $f\in\mathfrak{S}$.

Proof If $f\in\mathfrak{S}$, then $\mathfrak{U}=\mathfrak{M}f$ is a finite dimensional vector subspace of $\mathfrak{M}$ over Δ. Conversely it is easy to see that any finite dimensional vector subspace of $\mathfrak{M}$ over Δ has the form $\mathfrak{M}f$ where $f\in\mathfrak{S}$. Hence $\mathfrak{U}g=(\mathfrak{M}f)g=\mathfrak{M}(fg)=0$ if and only if $fg=0$. It follows that the set of neighborhoods of

$0\{(0;\mathfrak{U})\,|\,\mathfrak{U}$ a finite dimensional vector subspace of $\mathfrak{M}$ over $\Delta\}$

coincides with the set of $\mathfrak{Z}_R(f)$, $f\in\mathfrak{S}$.

3. Dual topology

Let $(\mathfrak{M}, \mathfrak{M}')$ be a pair of dual vector spaces over a division ring Δ. By means of natural isomorphism, $\mathfrak{M}'$ can be identified by a total subspace of the conjugate space $\mathfrak{M}^*$ of $\mathfrak{M}$. The finite topology in $\mathfrak{M}^*$ induces a topology in $\mathfrak{M}'$ by the requirement that the natural isomorphism of $\mathfrak{M}'$ into $\mathfrak{M}^*$ is a homeomorphism. This amount to that a base of neighborhoods of $0\in\mathfrak{M}'$ is the collection of sets

$$\{\mathfrak{U}^{\perp}\cap\mathfrak{M}'\,|\,\mathfrak{U} \text{ is a finite dimensional vector subspace of } \mathfrak{M}\},$$

$$\mathfrak{U}^{\perp}=\{y'\,|\,(x, y')=0 \text{ for all } x\in\mathfrak{U}\}.$$

We shall now denote $\mathfrak{U}^{\perp}\cap\mathfrak{M}'$ as $\mathfrak{U}^{\perp\mathfrak{M}'}$. We call the above topology the $\mathfrak{M}$-topology of $\mathfrak{M}'$. In a similar fashion we can identify $\mathfrak{M}$ with a subspace of the conjugate space $\mathfrak{M}'^*$ and define a topology in $\mathfrak{M}$, too. We call this topology

$$\{\mathfrak{U}'^{\perp \mathfrak{M}} \mid \mathfrak{U}' \text{ finite dimensional vector subspace of } \mathfrak{M}'\},$$

the $\mathfrak{M}'$-topology of $\mathfrak{M}$. Recall that a mapping A' of $\mathfrak{M}'$ is an adjoint of a linear transformation A of $\mathfrak{M}$ if and only if $(xA, y') = (x, y'A')$ for all $x \in \mathfrak{M}$, $y \in \mathfrak{M}'$. It follows immediately from the non-degeneracy of (x, y') that A' is a linear transformation of $\mathfrak{M}'$. It is well known that a linear transformation of $\mathfrak{M}$, topologized by $\mathfrak{M}'$-topology, is continuous if and only if A has an adjoint. And the ring of continuous linear transformations $\mathscr{L}_{\mathfrak{M}'}(\mathfrak{M})$ is anti-isomorphic $(A \to A')$ to the ring of continuous linear transformations $\mathscr{L}_{\mathfrak{M}}(\mathfrak{M}')$, i. e., the set of adjoints of elements in $\mathscr{L}_{\mathfrak{M}'}(\mathfrak{M})$.

THEOREM 3. *The ring $\mathfrak{A}$ can be topologized by the finite topology in $\mathfrak{A}'$ (the set of adjoints), and the base of neighborhoods of* 0 *is the collection $\mathfrak{Z}_L(f)$, $f \in \mathfrak{S}$.*

Proof Since the map $\eta: A \to A'$ is an anti-isomorphism, so is η^{-1}. Hence the base corresponding to the base $\mathfrak{Z}_R(f')$, $f' \in \mathfrak{S}'$, the finite topology in $\mathfrak{A}'$, are $\mathfrak{Z}_L(f)$, $f \in \mathfrak{S}$.

The new topology in $\mathfrak{A}$ given by the finite topology in $\mathfrak{A}'$ is called the **dual topology.**

4. Uniform topology

The $\mathfrak{M}'$-topology of $\mathfrak{M}$ was given by the base of neighborhoods of 0, $\{\mathfrak{U}'^{\perp \mathfrak{M}} \mid \mathfrak{U}'$ finite dimensional vector subspace of $\mathfrak{M}'\}$.

Now from this topology, another topology of $\mathfrak{A}$ will be considered, the **uniform topology** of $\mathfrak{A}$ acting on $\mathfrak{M}$. In this, a base of neighborhoods of 0 is given by the sets

$$\{B \in \mathfrak{A} \mid \mathfrak{M}B \subseteq \mathfrak{U}'^{\perp \mathfrak{M}}\}$$

THEOREM 5. *In the uniform topology of $\mathfrak{A}$, acting on $\mathfrak{M}$, the base of neighborhoods of* 0 *is $\mathfrak{Z}_L(f)$, $f \in \mathfrak{S}$. And in that of $\mathfrak{A}'$ acting on $\mathfrak{M}'$, the base of neighborhoods of* 0 *is $\mathfrak{Z}_L(f')$, $f' \in \mathfrak{S}'$.*

Proof. The sets given above is same as

$$\begin{aligned}
&\{B \in \mathfrak{A} \mid (xB, y') = 0 \text{ for all } x \in \mathfrak{M} \text{ and } y' \in \mathfrak{U}'\} \\
&= \{B \in \mathfrak{A} \mid x, y'B') = 0 \text{ for all } x \in \mathfrak{M} \text{ and } y' \in \mathfrak{U}'\} \\
&= \{B \in \mathfrak{A} \mid y'B' = 0 \text{ for all } y' \in \mathfrak{U}'\} \\
&= \{B \in \mathfrak{A} \mid \mathfrak{U}'B' = \mathfrak{M}'(f'B') = 0 \text{ for all } f' \in \mathfrak{S}'\} \\
&= \{B \in \mathfrak{A} \mid B' \in \mathfrak{Z}_R(f'),\ f' \in \mathfrak{S}'\} \\
&= \{\mathfrak{Z}_L(f) \mid f \in \mathfrak{S}\}.
\end{aligned}$$

This proves half the theorem. The other part is proved similarly.

References

1. N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloq. Publ., Vol. 37, 1968

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