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THE NIELSEN THEOREM FOR SEIFERT FIBERED SPACES OVER LOCALLY SYMMETRIC SPACES

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Abstract

In this note the geometric realization of a finite group of homotopy classes of self homotopy equivalences by a finite group of diffeomorphisms is investigated. In order for this to be accomplished an algebraic condition, which guarantees a certain group extension exists, must be satisfied. It is shown for a geometrically interesting class of aspherical manifolds, called injective Seifert fiber spaces over a locally symmetric space, that this necessary algebraic condition is also sufficient for geometric realization.

1. Introduction

For a closed oriented surface S of genus >0, any map $f: S\rightarrow S$ whose n-th power is homotopic to the identity can be homotoped to a diffeomorphism $F: S\rightarrow S$ whose n-th power is the identity. This famous result was first obtained by Nielsen [N], and later, an alternate proof, using the Smith theory, was found by Fenchel, [F]. Additional assumptions are needed to extend this result to bounded surfaces. Well known examples clearly show that extensions to higher dimensions must fail unless one imposes severe restrictions on the topology of the manifolds. Since the closed surfaces are aspherical, it would seem logical to look for extensions of the Nielsen-Fenchel result among the closed manifolds which are $K(\pi, 1)$'s. However, already for closed n-manifolds, $n>2$, which are aspherical, the Nielsen-Fenchel result surprisingly fails, [R-S]. All examples known to the author fail because a necessary algebraic condition is not satisfied. It would be very interesting to find a failure of the Nielsen-Fenchel result, on closed aspherical manifolds, without experiencing a failure of this algebraic condition. We will formulate this question explicitly later.

Let X be a reasonable type of space. Let $\mathcal{H}(X)$ be the group of self-homeomorphisms of X and $\mathcal{E}(X)$ the H-space of self homotopy equivalences. The inclusion $i: \mathcal{H}(X)\rightarrow\mathcal{E}(X)$ is an H-space homomorphism. There is a

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natural projection ψ from $\mathcal{E}(X)$ to the group of its path components $\pi_0(\mathcal{E}(X), 1_X)$. For reasonable spaces this group is countable and discrete (from the inherited topology). Moreover, if X is an aspherical space (i. e., a $K(\pi, 1)$) then $\pi_0(\mathcal{E}(X), 1_X)$ may be identified, by means of elementary obstruction theory, with $\operatorname{Out}\pi_1(X, *)$, where $\operatorname{Out}\pi$ denotes the group of automorphisms of π modulo inner automorphisms. We denote the composition $\psi\circ i$ by ψ_0. We formulate our problem now as follows:

PROBLEM. *Let $\theta: (F, e) \to \pi_0(\mathcal{E}(X), 1_X)$ be an injection of a finite group F. Does there exist a lifting $\bar{\theta}: G \to \mathcal{H}(X)$ so that $\bar{\psi}_0\circ\bar{\theta} = \bar{\theta}$?*

2. A group theoretic question

Since we are only interested in aspherical spaces, in this note, $\pi_0(\mathcal{E}(X), 1_X)$ can be replaced by $\operatorname{Out}(\pi_1(X, *))$, and $\bar{\theta}$ then is an *abstract kernel* in the sense used by Mac Lane in [M; *p.* 124]. In studying our problem we are immediately faced with a group theoretic question:

If a lifting $\bar{\theta}$ is to exist then the action of F on X gives rise to an action of E on the universal covering, $\tilde{X}$, of X. E is really the group of all liftings of F and is an extension:

$$1 \to \pi \to E \to F \to 1,$$

where $\pi = \pi_1(X, *)$ and F acts on π, via conjugation by elements of E, as the outer automorphisms $\theta(F)$. An explicit construction of this extension is given in [C-R-1; *p.* 5].

The first hurdle to be overcome, then, is to ascertain whether or not this algebraic extension exists. This question is well understood for there exists an obstruction class in $H^3(F;\ \text{center}(\pi))$ which must vanish, if and only if, such an extension is to exist. While it is not always easy to compute this obstruction, it can often be done. It is exactly here where the failure of generalizing Nielsen's theorem to higher dimensions has been detected. Therefore, the *existence* of this *extension* or what is the same the vanishing of our obstruction class, is a *necessary algebraic* condition that must be satisfied if we are going to successfully lift θ to $\mathcal{H}(X)$.

QUESTION (cf. [R-S]). *If X is a closed aspherical manifold, is the necessary algebraic condition also sufficient to lift θ to $\mathcal{H}(X)$?*

As mentioned earlier, no examples are known (to the author) where θ fails to have a lifting and the obstruction to an extension vanishes.

3. Seifert fibered spaces over locally symmetric spaces

In two earlier papers Conner and Raymond [C-R-2; 3] have shown that for certain geometrically interesting spaces an injection $\theta : F \to \mathrm{Out}\, \pi_1 (X, *)$ can be lifted to $\mathcal{H}(X)$. In the situations treated by them π has a normal subgroup K with N as *quotient* and the embedding θ induces the *trivial* outer automorphisms on N. For the results in [C-R-3; § 9] it is not necessary to assume X is aspherical and even the algebraic assumption of the existence of an extension is not made.

In this note we wish to deal with the *full outer automorphism group*. In [R-W], a class of aspherical manifolds called *injective Seifert fiber spaces with typical fiber a locally symmetric space* were constructed. It was shown that for those locally symmetric spaces called *infra-rigid manifolds* our problem always had an affirmative solution. The argument used Mostow's rigidity theorem.

We will use (implicitly) again Mostow's rigidity theorem, (but in a quite different way) in investigating our problem for a particular class of injective Seifert fiber spaces with typical fiber a k-torus. We shall describe this particular class now.

We recall, [R-W], that a lattice Γ is called *rigid* if it is a lattice in a connected, semi-simple Lie group, G, without compact factors and such that the projection of Γ to any 3-dimensional factor is dense. A lattice Γ_1 in Aut $G=\bar{G}$ is called *infra-rigid* if $\Gamma=\Gamma_1 \cap G$ is *rigid*. Aut G is a finite extension (by Out G) of G. In fact. it is a semi-direct product $G \rtimes \mathrm{Out}\, G$ and is centerless. Let K be a maximal compact subgroup of G and $\bar{K}$, $K \subset \bar{K}$, a maximal compact subgroup of $\bar{G}$, $\bar{K}\backslash\bar{G}$ is naturally isomorphic to $\bar{K}\backslash G$ which is diffeomorphic to R^n, for some n. We denote Γ_1 by N and $\bar{K}\backslash\bar{G}$ by W to conform with the notation of [C-R-3]. Let $\varphi : N \to GL(k, \mathbf{Z})$ be a homomorphism, then the elements $a \in H^2_{\varphi}(N:\mathbf{Z}^k)$ classify the group extensions:

$$a : 1 \to \mathbf{Z}^k \to \pi \to N \to 1,$$

where N operates on $\mathbf{Z}^k$ by way of φ. Since N operates properly discontinuously on the right of W by $(\bar{K}g) \times n \to \bar{K}gn$, the orbit space W/N is the space of double cosets $\bar{K}\backslash\bar{G}/N$. We may use this space as a parameter space and construct for each extension $a \in \mathrm{H}^2 (N; \mathbf{Z}^k)$ a Seifert fiber space $M(a)$ over it. We refer the reader to [C-R-4 or 3] for the details.

In fact, corresponding to each "a" we may, using the N action on W, construct a unique, up to equivalence, (right) action of the semi-direct product $T^k \rtimes_{\varphi} N$ on $T^k \times W$. The T^k action is just translation on the first factor and the projection of $T^k \times W$ onto W is equivariant with respect to the action of $N(=1\times N)$ on $T^k\times W$ and the given N action on W. The action of the not necessarily normal subgroup N on $T^k\times W$ is properly discontinuous and *is a covering action, if and only if, the group* π *is torsion free.* The set $B_\varphi \subset H^2{}_\varphi(N;Z^k)$ of such torsion free extensions is called the *set of Bieberbach classes.* If $a\in B_\varphi$ then $M(a)=(T^k\times W)/N$ is called an *injective Seifert fiber space over the locally symmetric space,* W/N. In our definition it certainly is possible for W/N to be singular in the sense of failing to be locally Euclidean. There is a natural map μ of $M(a)$ onto $W/N=\bar{K}\backslash\bar{G}/N$ so that each $\mu^{-1}(w^*)$ is a flat Riemannian manifold isomorphic to T^k/N_w, where $w\to w^*\in W/N$. The induced extension.

$$i_w{}^*(a) : 1\to \mathbf{Z}^k\to\pi_1(\mu^{-1}(w^*))\to N_w\to 1$$

parameterizes the fundamental groups of the fibers. Here i_w is the inclusion of the stabilizer N_w into N. Our $M(a)$'s have a natural smooth structure. (We remark that N_w is determined by the torsion in Γ_1. For, if w_0 is the coset $\bar{K}$, and w is the coset $\bar{K}g$ then $(\Gamma_1)_w=N_w$ is the finite subgroup $\Gamma_1\cap g^{-1}\bar{K}g$).

The following Proposition is proved in [C-R-3: § 8].

PROPOSITION. *The mapping* $\psi_0 : \mathcal{H}(M(a))\to Out\ (\pi_1(M(a)))$ *is onto.*

In fact the subgroup, $\mathcal{A}(M(a))$, of *automorphisms of the Seifert fiber space* (what is called *the fiber preserving homeomorphisms of the Seifert fibering* in [C-R-3]) is mapped surjectively. Therefore, it will be a great geometric advantage to attempt to solve our problem by lifting to $\mathcal{A}(M(a))$ instead of $\mathcal{H}(M(a))$. This is what we shall do now for *injective Seifert fiber spaces* $M(a)$ *over a locally symmetric space* W/N.

4. Geometric realization

Let $\theta : F\to \mathrm{Out}(\pi_1(M(a)))=\pi$ be an injection of a finite group. This defines an abstract kernel and suppose there is some extension:

$$1\to\pi\to E\to F\to 1$$

realizing this abstract kernel.
(This can be thought of as a cohomology class $\theta(a)$ in $H^2(F;\ \text{center}\ \pi)$).

THEOREM. θ *lifts to* $\mathcal{A}(M(a))$.

Proof. Because N is an *infra-rigid* lattice, N is centerless. Also, the normal subgroup $\mathbf{Z}^k$ in π, from the extension class a, is a characteristic subgroup by 6.2 of [C-R-3]. Consequently, this group, as a subgroup of the extension E, is a normal subgroup of E. Therefore, we may induce the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 1 & \longrightarrow & 1 & & & & \\
 & & \downarrow & = & \downarrow & & & & \\
1 & \longrightarrow & \mathbf{Z}^k & \longrightarrow & \mathbf{Z}^k & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi & \longrightarrow & E & \longrightarrow & F & \longrightarrow & 1\\
 & & \downarrow & j & \downarrow & & \vdots & & \\
1 & \longrightarrow & N & \dashrightarrow & E/\mathbf{Z}^k & \longrightarrow & F & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

The dotted lines indicate naturally induced homomorphisms. The third horizontal row is induced by:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & L=E/\mathbf{Z}^k & \longrightarrow & F & \longrightarrow & 1\\
 & & \downarrow \theta''' & & \downarrow \theta'' & & \downarrow \theta' & & \\
1 & \rightarrow & N=\mathrm{Inn}N & \rightarrow & \mathrm{Aut}N & \longrightarrow & \mathrm{Out}N & \rightarrow & 1,
\end{array}$$

where $\theta'(f)$ is the outer automorphism induced by the action of E on π. Now, θ' could have non-trivial kernel but this will make no difference. We are first going to geometrically realize the action of L on W by just letting it act via its image $\theta''(L)\subset\mathrm{Aut}\ N$. Aut N does act ***properly discontinuously*** on $\bar{W}=\bar{G}\backslash\bar{K}$ by [R-W]. (The kernel of θ''=the kernel of θ'). It is exactly here where we use the strong conditions on (W, N).

Now, the extension

$$a' : 1 \longrightarrow \mathbf{Z}^k \longrightarrow E \longrightarrow L \longrightarrow 1$$

is given by $a'\in H^2_\varphi(L;\mathbf{Z}^k)$ and the inclusion $j : N\subset L$ induces the extension class $a=j^*(a')$. Therefore, using the properly discontinuous action (W, L) and a' we may construct, as mentioned earlier, an action of $T^k\rtimes_{\varphi'}L$ on $T^k\times W$ which restricts to an $T^k\rtimes_\varphi N$ action equivalent to the original action determined by a. Consequently, dividing out first by N, we obtain $M(a)$. Then, by dividing out by $L/N=F$, we obtain our desired F-action as a subgroup of the group, $\mathcal{A}(M(a))$, of automorphisms of the aspherical injective Seifert fiber space $M(a)$ over W/N. This completes the proof.

REMARKS. Our construction of the actions $T^k\rtimes_\varphi N$ and $T^k\rtimes_{\varphi'}L$ can be done smoothly. So our geometric realization is actually in the group of smooth automorphisms of the smooth injective Seifert fiber space $M(a)$.

If the extension E is torsion free, then the action of F on $M(a)$ is free and $M(a)/F=M(a')$ will also be an aspherical injective Seifert fiber space over W/L.

The manifold $M(a)$ is closed, if and only if, N is uniform in $\bar{G}$; however finite volume of $\bar{G}/N$ suffices for our result.

We should also observe that if the abstract kernel θ can be realized by some extension E, then all the possible extensions realizing this kernel are given by $H^2(F;\ \text{center}(\pi))$. All of these various extensions must give rise to various liftings of θ to $\mathcal{A}(M(a))$. However, since N is centerless and $\theta'(F)$ is determined by the projection of $\text{Out}(\pi)\to\text{Out}(N)$, we see that the abstract kernel θ determines L and its action (W, L) up to equivalence. It is still possible that various extensions give rise to non-equivalent F actions on $M(a)$.

One of the most interesting situations not covered by our theorem, is the case of $G=PSL(2, R)$ and N a Fuchsian group. (If $k=1$, this yields the classical Seifert 3-manifolds.) The problem encountered is that one does not always know whether the induced extension L of N by F is a Fuchsian group. In this 3-dimensional case almost all injective θ give rise to injective θ'. If it is known that L is Fuchsian, then we obtain the same result as our Theorem. This case will be discussed in detail in a forthcoming paper by W. D. Neumann and the author, "Automorphisms of Seifert manifolds." The classical 3-dimensional case has also been investigated, especially for involutions, from a different point of view by Tolleffson [T] and Heil-Tolleffson [H-T]. There they go a step further and not only show for most instances that one can realize involutions when the algebraic obstruction vanishes but all smooth involutions are equivalent to involutions in $\mathcal{A}(M(a))$.

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