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INFINITESIMAL VARIATIONS PRESERVING C-BOCHNER CURVATURE TENSOR OF AN INVARIANT SUBMANIFOLD OF A SASAKIAN MANIFOLD

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§0. Introduction

Recently infinitesimal variations of invariant submanifolds of a Sasakian manifold have been studied by Yano, Ki and Pak [4]. They proved the following:

THEOREM A [4]. *If an infinitesimal isometric invariant variation of a compact orientable invariant submanifold of a Sasakian manifold is fiber-preserving, then it is f-preserving.*

Since invariant submanifolds of a Sasakian manifold induce a Sasakian structure, we may consider C-Bochner curvature tensor (or contact Bochner curvature tensor) on the submanifold.

In the present paper we study infinitesimal variations preserving the C-Bochner curvature tensor of an invariant submanifold of a Sasakian manifold, and prove Theorem 3.2 (see §3) by using Theorem A.

§1. Sasakian submanifolds of a Sasakian manifold

Let M^{2m+1} be a $(2m+1)$-dimensional Sasakian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and let M^{2m+1} admits the set of structure tensors $(f_i^h,\ g_{ji},\ f_i)$ of M^{2m+1}, where here and in the sequel, the indices $h, i, j, \cdots$ run over the range $\{1', 2', \cdots, (2m+1)'\}$. Then we have

$$f_i^t f_t^h = -\delta_i^h + f_i f^h,\quad f_t f_i^t = 0,\quad f_t^h f^t = 0, \tag{1.1}$$
$$f_t f^t = 1,\quad f_j^t f_i^s g_{ts} = g_{ji} - f_j f_i$$

where $f^h = f_i g^{ih}$, and

$$\nabla_j f^h = f_j^h,\quad \nabla_j f_i^h = -g_{ji} f^h + \delta_j^h f_i \tag{1.2}$$

∇_i denoting the operator of covariant differentiation with respect to g_{ji}. Let $M^{2n+1}(n<m)$ be a $(2n+1)$-dimensional Riemanian manifold covered by a

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system of coordinate neighborhoods $\{V;\ y^a\}$ and isometrically immersed in M^{2m+1} by the immersion $i: M^{2n+1} \rightarrow M^{2m+1}$, where here and in the sequel, the indices $a, b, c, \cdots$ run over the range $\{1, 2, \cdots, (2n+1)\}$. We identify $i(M^{2n+1})$ with M^{2n+1} and represent the immersion by $x^h = x^h(y^a)$. If we put $B_b{}^h = \partial_b x^h$, $\partial_b = \partial/\partial y^b$, then $B_b{}^h$ are $2n+1$ linearly independent vectors of M^{2m+1}. Denoting by g_{cb} the Riemannian metric tensor of M^{2n+1}, we have $g_{cb} = g_{ji} B_c{}^j B_b{}^i$ since the immersion is isometric. We denote by $C_y{}^h$ $2(m-n)$ mutually orthogonal unit normals to M^{2n+1}, then we have $g_{ji} B_b{}^j C_y{}^i = 0$ and the metric tensor of the normal bundle of M^{2n+1} is given by $g_{zy} = g_{ji} C_z{}^j C_y{}^i = \delta_{zy}$, δ_{zy} being the Kronecker delta, where here and in the sequel, the indices u, v, x, y, z run over the range $\{(2n+2), \cdots, (2m+1)\}$.

We denote by Γ_{ji}^h and Γ_{cb}^a the Christoffel symbols formed with g_{ji} and those formed with g_{cb} respectively and denote by $\Gamma_{cy}{}^x$ the components of the connection induced in the normal bundle of M^{2n+1}, that is

$$(1.3)\qquad \Gamma_{cy}^x = (\partial_c C_y{}^h + \Gamma_{ji}^h B_c{}^j C_y{}^i) C_h{}^x,$$

$C_h{}^x = C_y{}^i g^{yx} g_{ih}$, g^{yx} being the contravariant components of the metric tensor of the normal bundle. Then the van der Waerden-Bortolotti covariant derivatives of $B_b{}^h$ and $C_y{}^h$ are given by

$$\nabla_c B_b{}^h = \partial_c B_b{}^h + \Gamma_{ji}^h B_c{}^j B_b{}^i - \Gamma_{cb}^a B_a{}^h$$

and

$$\nabla_c C_y{}^h = \partial_c C_y{}^h + \Gamma_{ji}^h B_c{}^j C_y{}^i - \Gamma_{cy}^x C_x{}^h$$

respectively and the equations of Gauss and Weingarten are respectively

$$(1.4)\qquad \nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h,\quad \nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h,$$

where $h_{cb}{}^x$ are components of the second fundamental tensors of M^{2n+1} and $h_c{}^a{}_y = h_{cb}{}^x g^{ba} g_{xy}$, g^{ba} being components of the metric tensor of M^{2n+1}.

A $(2n+1)$-dimensional submanifold M^{2n+1} is called an invariant submanifold of the Sasakian manifold M^{2m+1} if the tangent space at each point of M^{2n+1} is invariant under the action of $f_i{}^h$. Thus for an invariant submanifold M^{2n+1}, we have

$$(1.5)\qquad f_i{}^j B_b{}^i = f_b{}^a B_a{}^h,\quad f_i{}^h C_y{}^i = f_y{}^x C_x{}^h,$$

$f_b{}^a$ and $f_y{}^x$ being tensor fields of type $(1,1)$ of M^{2n+1} and of the normal bundle of M^{2n+1} respectively. Putting $f_{ba} = f_b{}^e g_{ea}$ and $f_{yx} = f_y{}^z g_{zx}$, we have $f_{ba} = f_{ji} B_b{}^j B_a{}^i$ and $f_{yx} = f_{ji} C_y{}^j C_x{}^i$ and consequently

$$(1.6)\qquad f_{ba} = -f_{ab},\quad f_{yx} = -f_{xy}.$$

On the other hand, putting

(1.7) $$f^h = f^a B_a{}^h + f^x C_x{}^h,$$

we easily see from (1.5) and (1.7) that

(1.8) $$f_b{}^e f_e{}^a = -\delta_b^a + f_b f^a, \quad f_b f^x = 0,$$

which shows that $f^x = 0$([4]). Thus (1.7) becomes

(1.9) $$f^h = f^a B_a{}^h.$$

From (1.5) and (1.9) we have the followings:

(1.10) $$f_b{}^e f_e{}^a = -\delta_b^a + f_b f^a, \quad f_e{}^a f^e = 0, \quad f_e f_c{}^e = 0,$$
$$f_e f^e = 1, \quad g_{ed} f_c{}^e f_b{}^d = g_{cb} - f_c f_b$$

and

(1.11) $$f_y{}^z f_z{}^x = -\delta_y^x, \quad f_z{}^v f_y{}^u g_{vu} = g_{zy}.$$

The equations (1.10) shows that the invariant submanifold M^{2n+1} admits an almost contact metric structure ($f_b{}^a$, g_{cb}, f_a) and (1.11) mean that $f_y{}^x$ defines an almost Hermitian structure in the normal bundle.

Now differentiating (1.5) and (1.9) covariantly along M^{2n+1} and using (1.2), (1.4), (1.5) and (1.9), we find respectively

(1.12) $$\nabla_c f_b{}^a = -g_{cb} f^a + \delta_c^a f_b, \quad \nabla_c f_b = f_{cb},$$

(1.13) $$h_{cb}{}^y f_y{}^x = h_{ce}{}^x f_b{}^e, \quad h_{cb}{}^x f^b = 0,$$

(1.14) $$\nabla_c f_y{}^x = 0.$$

The equation (1.12) shows that the almost contact metric structure on the invariant submanifold M^{2n+1} is a Sasakian structure. On a Sasakian manifold, it is well known that the following identites are valid:

(1.15) $$K_{be} f_a{}^e + K_{ae} f_b{}^e = 0, \quad K_{be} f^e = 2n f_b,$$

where $K_{cb}(=K_{dcb}{}^d)$ is the Ricci tensor and $K_{dcb}{}^a$ the curvature tensor of M^{2n+1}.

The components $B_{dcb}{}^a$ of the C–Bochner curvature tensor in a Sasakian manifold M^{2n+1} are given by [2]:

(1.16) $$B_{dcb}{}^a = K_{dcb}{}^a + \frac{1}{2n+4}(K_{db}\delta_c^a - K_{cb}\delta_d^a + g_{db}K_c{}^a - g_{cb}K_d{}^a + S_{db}f_c{}^a$$
$$- S_{cb}f_d{}^a + f_{db}S_c{}^a - f_{cb}S_d{}^a + 2S_{dc}f_b{}^a + 2f_{dc}S_b{}^a - K_{db}f_c f^a + K_{cb}f_d f^a$$
$$- f_d f_b K_c{}^a + f_c f_b K_d{}^a) - \frac{k+2n}{2n+4}(f_{db}f_c{}^a - f_{cb}f_d{}^a + 2f_{dc}f_b{}^a)$$

$$-\frac{k-4}{2n+5}(g_{db}\delta_c^a - g_{cb}\delta_d^a) + \frac{k}{2n+4}(g_{db}f_c f^a + f_d f_b \delta_c^a - g_{cb} f_d f^a - f_c f_b \delta_d^a),$$

where $S_{cb}=f_c{}^e K_{eb}$, $S_c{}^b = S_{ce}g^{be}$, $K_c{}^a = K_{cb}g^{ba}$ and $k=\frac{K+2n}{2n+2}$, denoting by K the scalar curvature of M^{2n+1}.

From (1.16), we see that

(1.17) $$B_{dcb}{}^d = 0.$$

Transvecting (1.16) with K^{cb} and using (1.15) and (1.17), we easily find that

(1.18) $$K^{cb}B_{dcb}{}^a = K^{cb}K_{dcb}{}^a + \frac{1}{2n+4}(2nK_d{}^a - KK_d{}^a - 4K_d{}^e K_e{}^a - K^{cb}K_{cb}\delta_d^a + 16n^2 f_d f^a + K_{cb}K^{cb} f_d f^a) - \frac{3(k+2n)}{2n+4}(K_d{}^a - 2nf_d f^a) + \frac{k-4}{2n+4}(K_d{}^a - K\delta_d^a) - \frac{k}{2n+4}(4nf_d f^a - Kf_d f^a - 2n\delta_d^a).$$

where $K^{cb} = K_{de}g^{dc}g^{eb}$.

§2. Infinitesimal variations of a Sasakian submanifold

Let M^{2n+1} be a $(2n+1)$-dimensional invariant submanifold of a Sasakian manifold M^{2m+1}, and consider an infinitesimal variation given by

(2.1) $$\bar{x}^h = x^h + v^h(y)\varepsilon,$$

where ε is an infinitesimal. Putting $\bar{B}_b{}^h = \partial_b \bar{x}^h$, we have

(2.2) $$\bar{B}_b{}^h = B_b{}^h + (\partial_b v^h)\varepsilon.$$

The $(2n+1)$-linearly independent vectors $\bar{B}_b{}^h$ are tangent to the varied submanifold at $(\bar{x}^h)$. We displace $\bar{B}_b{}^h$ back parallelly from $(\bar{x}^h)$ to (x^h), then we have

(2.3) $$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}^h(x+v\varepsilon)v^j \bar{B}_b{}^i \varepsilon.$$

Thus, putting $\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h$, we find

(2.4) $$\delta B_b{}^h = (\nabla_b v^h)\varepsilon,$$

neglecting terms of order higher than one with respect to ε, where $\nabla_b v^h = \partial_b v^h + \Gamma_{ji}^h B_b{}^j v^i$.

In the sequel we always neglect terms of order higher than one with respect to ε.

On the other hand, putting $v^h = v^a B_a{}^h + v^x C_x{}^h$, we then have

$$\nabla_b v^h = (\nabla_b v^a - h_b{}^a{}_x v^x) B_a{}^h + (\nabla_b v^x + h_{ba}{}^x v^a) C_x{}^h. \tag{2.5}$$

Thus (2.4) can be rewritten as

$$\delta B_b{}^h = [(\nabla_b v^a - h_b{}^a{}_x v^x) B_a{}^h + (\nabla_b v^x + h_{ba}{}^x v^a) C_x{}^h]\varepsilon. \tag{2.6}$$

An infinitesimal variation (2.1) carries an invariant submanifold into an invariant one is called an *invariant variation* ([3], [4]).

For an invariant submanifold M^{2n+1} of a Sasakian manifold M^{2m+1}, Yano K., U-Hang Ki and Jin Suk Pak have proved the following

THEOREM 2.1([4]). *In order for an infinitesimal variation of an invariant submanifold M^{2n+1} to be invariant it is necessary and sufficient that*

$$(\nabla_b v^y) f_y{}^x - f_b{}^e \nabla_e v^x + f_b v^x = 0. \tag{2.7}$$

In this case the variations of the structure tensors of M^{2n+1} are given by

$$\delta f_b{}^a = [f_e{}^a (\nabla_b v^e - h_b{}^e{}_x v^x) - f_b{}^e (\nabla_e v^a - h_e{}^a{}_x v^x) + (f_b v^a - v_b f^a)]\varepsilon, \tag{2.8}$$

$$\delta f^a = [\mathcal{L} f^a + \alpha f^a]\varepsilon, \tag{2.9}$$

$\mathcal{L}$ denoting Lie derivation with respect to v^a.

When an infinitesimal variation satisfies $\delta f_b{}^a = 0$, we say that the invariant variation is *f-preserving* ([4]).

Now appling the operator δ to $g_{cb} = g_{ji} B_c{}^j B_b{}^i$ and using (2.6) and the fact that $\delta g_{ji} = 0$, we find

$$\delta g_{cb} = [\nabla_c v_b + \nabla_b v_c - 2h_{cbx} v^x]\varepsilon, \tag{2.10}$$

from which

$$\delta g^{ba} = -[\nabla^b v^a + \nabla^a v^b - 2h^{ba}{}_x v^x]\varepsilon, \tag{2.11}$$

where $\nabla^b = g^{ba} \nabla_a$ and $h^{ba}{}_x = g^{be} g^{ad} h_{edx}$.

An infinitesimal variation for which $\delta g_{cb} = 0$ is said to be *isometric* ([3]).

Moreover, it is well known (cf. [5]) that the variations of the Christoffel symbols Γ^a_{cb} formed with g_{cb} are given by

$$\delta \Gamma^a_{cb} = (\nabla_c \nabla_b v^a + K_{dcb}{}^a v^d)\varepsilon - [\nabla_c (h_{bex} v^x) + \nabla_b (h_{cex} v^x) - \nabla_e (h_{cbx} v^x)] g^{ea}\varepsilon, \tag{2.12}$$

and the variation of the curvature tensor $K_{dcb}{}^a$ by

$$\delta K_{dcb}{}^a = \nabla_d (\delta \Gamma^a_{cb}) - \nabla_c (\delta \Gamma^a_{db}). \tag{2.13}$$

We assume that the invariant variation of the invariant submanifold is isometric. Then we can find that ([4]) $\delta\Gamma^a_{cb}=0$, from which

$$\delta K_{dcb}{}^a=0 \tag{2.14}$$

and consequently

$$\delta K_{cb}=0,\ \delta K^{cb}=0,\ \delta K_c{}^b=0,\ \delta K=0. \tag{2.15}$$

§3. Infinitesimal variations preserving C-Bochner curvature

We first suppose that an infinitesimal isometric invariant variation of an invariant submanifold M^{2n+1} is f-preserving, that is, $\delta f_b{}^a=0$.

Applying the operator δ to $f_c f^c=1$, we have by the isometricity of the variation

$$f_c\delta f^c=0. \tag{3.1}$$

On the other hand, applying the operator δ to $f_c{}^a f^c=0.$, we find

$$f_c{}^a\delta f^c=0. \tag{3.2}$$

Transvecting $f_a{}^b$ to (3.2) and taking account of (1.8), we have

$$\delta f^c=(f_b\delta f^b)f^c, \tag{3.3}$$

from which, using (3.1), we find

$$\delta f^c=0. \tag{3.4}$$

Now applying the operator δ to (1.16), using $\delta f_b{}^a=0$, (2.14), (2.15) and (3.4), we have

$$\delta B_{dcb}{}^a=0. \tag{3.5}$$

When an infinitesimal invariant variation satisfies $\delta B_{dcb}{}^a=0$, we say that the invariant variation is *C-Bochner preserving*. Thus, from (3.5), we have the following

PROPOSITION 3.1. *If an infinitesimal isometric invariant variation of an invariant submanifold of a Sasakian manifold is f-preserving, then it is C-Bochner preserving.*

Next we consider converse problem of above proposition.

Applying the operator δ to (1.18) and using (2.14) and (2.15), we find

$$K^{cb}\delta B_{dcb}{}^a=A\delta(f_d f^a), \tag{3.6}$$

where we have put $A=\frac{1}{2(n+2)}[4n^2-k(K+2n)+K_{cb}K^{cb}]$.

Since

$$(3.7)\quad K_{cb}K^{cb}-\frac{(K+2n)^2}{2n+2}+4n^2=\|K_{cb}-\frac{K}{2n+1}g_{cb}\|^2+\frac{[K-2n(2n+1)]^2}{2(2n+1)(n+1)},$$

we see that A in (3.6) is always positive if $K\neq 2n(2n+1)$.

Hence we have

$$(3.8)\qquad \delta(f_d f^a)=0$$

under the action of the C-Bochner preserving variation if $K\neq 2n(2n+1)$ and (3.8) implies that

$$(3.9)\qquad \delta f^a=0.$$

An infinitesimal variation which satisfies $\alpha f^a=\beta f^a$, β being a certain function, is said to be fiber-preserving [4]. Therefore taking account of (2.9) and (3.9), we can see that the C-Bochner preserving variation is fiber-preserving. Thus, by the help of Theorem A, we have the following

THEOREM 3.2. *If an infinitesimal isometric invariant variation of a compact orientable invariant submanifold of a Sasakian manifold is C-Bochner preserving and if the scalar curvature K satisfies* $K\neq 2n(2n+1)$, *then it is f-preserving.*

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