J. Korean Math. Soc.
Vol. 16, No. 1, 1979

ON REAL HYPERSURFACES OF A QUATERNIONIC PROJECTIVE SPACE

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1. Introduction

As a characterization of Quaternionic Kaehlerian manifold $QP(m)$ of real dimension $4m$ $(m\geq 2)$ with constant Q-sectional curvature c, we gave the existence of a real hypersurface M, which satisfies the condition

$$(1.1)\qquad A(X,\ Y)=\frac{c}{4}g(X,\ Y)-\{u(X)u(Y)+v(X)v(Y)+w(X)w(Y)\},$$

passing through an arbitrary point and being tangent to an arbitrary $(4m-1)$-direction at that point, where g and A denote the first and second fundamental tensors of M respectively and u, v, w some local 1-forms defined by (2.3) ([1]). For the hypersurface M we can easily see that its Ricci tensor R_0 satisfies

$$(1.2)\qquad R_0(X,\ Y)=\alpha g(X,\ Y)+\beta\{u(X)u(Y)+v(X)v(Y)+w(X)w(Y)\}$$

by using the Gauss equation of the immersion $i:M\rightarrow QP(m)$ and (1.1), where α and β are absolute constants concerned with c.

In general, if the Ricci tensor R_0 of a real hypersurface M of $QP(m)$ satisfies (1.2) for some differentiable functions α and β, we call M a *Q-Einstein hypersurface*. When $\beta=0$, M is an Einstein space.

In this paper we shall consider a Q-Einstein hypersurface. In order to determine a Q-Einstein hypersurface we first introduce a theorem which has been proved by one of the present authors ([6]):

THEOREM A. *Let M be a real hypersurface of a quaternionic projective space $QP(m)$ and $\pi:\overline{M}\rightarrow M$ the submersion which is compatible with the Hopf fibration $S^{4m+3}(1)\rightarrow QP(m)$. Suppose one of the following conditions* (1)～(3) *which are equivalent to each other is valid:*

(1) *The induced almost contact 3-structure tensors $\{\phi, \psi, \theta\}$ in M commute with its second fundamental tensor.*

(2) *The induced almost contact 3-structure in M is normal.*

(3) $g((\nabla_Z A)X,\ Y)=-u(X)g(\phi Z,\ Y)-u(Y)g(\phi Z,\ X)-v(X)g(\psi Z,\ Y)$
$-v(Y)g(\psi Z,\ X)-w(X)g(\theta Z,\ Y)-w(Y)g(\theta Z,\ X).$

Received May 1, 1979

Then $M=M^Q_{p,q}(a,b)$ *for some portion* (p,q) *of* $m-1$ *and some* a, b *such that* $a^2+b^2=1$.

The model space $M^Q_{p,q}(a,b)$ in the above theorem is described in the following:

We denote by $S^{4p+3}(a)$ the hypersphere of radius a centered at the origin in a $(p+1)$-dimensional space Q^{p+1} of quaternions, which will be identified naturally with $R^{4(p+1)}$. If we identify Q^{p+q+2} with the product space $Q^{p+1}\times Q^{q+1}$, then, taking spheres $S^{4p+3}(a)$ in Q^{p+1} and $S^{4q+3}(b)$ in Q^{q+1}, we consider the product space $\overline{M}^Q_{p,q}(a,b)=S^{4p+3}(a)\times S^{4q+3}(b)$, which is naturally considered as a submanifold in Q^{p+q+2}. When $a^2+b^2=1$, $\overline{M}^Q_{p,q}(a,b)$ is a hypersurface in $S^{4(p+q+1)+3}(1)\subset Q^{p+q+2}$. Thus, if $a^2+b^2=1$, for any portion (p,q) of an integer $m-1$ such that $p+q=m-1$, $p\geqq 0$, $q\geqq 0$, $\overline{M}^Q_{p,q}(a,b)$ may be considered as a real hypersurface of $S^{4m+3}(1)\subset Q^{m+1}$.

Let $\tilde{\pi}:S^{4m+3}(1)\to QP(m)$ be the natural projection of S^{4m+3} onto a quaternionic projective space $QP(m)$ which is defined by the Hopf fibration. As is well known, the base space $QP(m)$ of a fibred Riemannian space with Sasakian 3-structure admits the induced a quaternionic Kaehlerian structure, and moreover is of constant Q-sectional curvature 4 ([2], [6]).

Considering the Hopf fibration $\tilde{\pi}:S^{4m+3}(1)\to QP(m)$, we put $M^Q_{p,q}(a,b)=\tilde{\pi}(\overline{M}^Q_{p,q}(a,b))$, which gives a Riemannian submersion

$$\pi:\overline{M}^Q_{p,q}(a,b)\to M^Q_{p,q}(a,b)$$

compatible with the Hopf fibration $\tilde{\pi}$, that is, a Riemannian submersion with totally geodesic fibres such that the following diagram commutes:

$$\begin{array}{ccc}\overline{M}_{p,q}(a,b) & \longrightarrow & S^{4m+3}(1)\\ \pi\downarrow & & \downarrow\tilde{\pi}\\ M^Q_{p,q}(a,b) & \longrightarrow & QP(m).\end{array}$$

2. Hypersurfaces of a quaternionic Kaehlerian manifold with constant Q-sectional curvature

We first recall the definition of a quaternionic Kaehlerian structure given by S. Ishihara ([2]). Let $\overline{M}$ be a $4m$-dimensional differentiable manifold and assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over $\overline{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\{\bar{U};\ y^h\}$, there is a local base {F, G, H} of V such that

$$
\begin{aligned}
&F_h{}^i F_j{}^h = -\delta_j^i,\quad G_h{}^i G_j{}^h = -\delta_j^i,\quad H_h{}^i H_j{}^h = -\delta_j^i,\\
(2.1)\qquad &F_h{}^i G_j{}^h = -G_h{}^i F_j{}^h = H_j{}^i,\quad G_h{}^i H_j{}^h = -H_h{}^i G_j{}^h = F_j{}^i,\\
&H_h{}^i F_j{}^h = -F_h{}^i H_j{}^h = G_j{}^i,
\end{aligned}
$$

$F_j{}^i$, $G_j{}^i$ and $H_j{}^i$ denoting components of F, G and H in $\bar{U}$ respectively.

(b) There is a Riemannian metric tensor g_{ji} such that

$$F_{ji} = -F_{ij},\quad G_{ji} = -G_{ij},\quad H_{ji} = -H_{ij},$$

where $F_{ji} = g_{hi} F_j{}^h$, $G_{ji} = g_{hi} G_j{}^h$ and $H_{ji} = g_{hi} H_j{}^h$.

(c) For the Riemannian connection $\bar{\nabla}$ of $(\bar{M}, g)$

$$
\begin{aligned}
&\bar{\nabla}_j F_i{}^h = r_j G_i{}^h - q_j H_i{}^h,\\
(2.2)\qquad &\bar{\nabla}_j G_i{}^h = -r_j F_i{}^h + p_j H_i{}^h,\\
&\bar{\nabla}_j H_i{}^h = q_j F_i{}^h - p_j G_i{}^h,
\end{aligned}
$$

where $p = p_i dy^i$, $q = q_i dy^i$ and $r = r_i dy^i$ are certain local 1-forms defined in $\bar{U}$. Such a local base $\{F, G, H\}$ is called a canonical local base of the bundle V in $\bar{U}$, and $(\bar{M}, g, V)$ or $\bar{M}$ is called a quaternionic Kaehlerian manifold and (g, V) a quaternionic Kaehlerian structure.

In a quaternionic Kaehlerian manifold $(\bar{M}, g, V)$ we take intersecting coordinate neighborhoods $\bar{U}$ and $'\bar{U}$. Let $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ be canonical local bases of V in $\bar{U}$ and $'\bar{U}$ respectively. Then it follows that in $\bar{U} \cap {}'\bar{U}$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix}, \qquad (x, y = 1, 2, 3)$$

with differentiable function s_{xy}, where the matrix $s = (s_{xy})$ is contained in the special orthogonal group $SO(3)$ as a consequence of (2.1). As is well known, a quaternionic Kaehlerian manifold is orientable.

We consider a real hypersurface M in a quaternionic Kaehlerian manifold $\bar{M}$ of dimension $4m$. Let $\bar{M}$ is covered by a system of coordinate neighborhoods $\{\bar{U};\ y^h\}$. Then M is covered by a system of coordinate neighborhoods $\{U;\ y^a\}$, where $U = \bar{U} \cap M$. Let M be represented by $y^i = y^i(x^a)$ with respect to local coordinates (y^i) in $\bar{U}\ (\subset \bar{M})$ and (y^a) in $U\ (\subset M)$. Denoting the vectors $\partial_a y^i$ $(\partial_a = \partial/\partial y^a)$ tangent to M by $B_a{}^i$ and a unit normal vector field by N^i, we can put in each coordinate neighborhood $U =$

$\bar{U}\cap M$

$$\text{(2.3)}\quad \begin{aligned} &\text{(i)}\ F_h{}^iB_a{}^h=\phi_a{}^bB_b{}^i+u_aN^i,\ \ F_h{}^iN^h=-u^aB_a{}^i,\\ &\text{(ii)}\ G_h{}^iB_a{}^h=\psi_a{}^bB_b{}^i+v_aN^i,\ \ G_h{}^iN^h=-v^aB_a{}^i,\\ &\text{(iii)}\ H_h{}^iB_a{}^h=\theta_a{}^bB_b{}^i+w_aN^i,\ \ H_h{}^iN^h=-w^aB_a{}^i,\end{aligned}$$

$\phi_a{}^b, \psi_a{}^b, \theta_a{}^b$ being local tensor fields of type (1,1) and u_a, v_a, w_a local 1-forms defined in U, where $g_{ba}=g_{ji}B_b{}^jB_a{}^i$ are the components of the induced metric tensor in M. We have easily $u^b=g^{ba}u_a$, $v^b=g^{ba}v_a$, $w^b=g^{ba}w_a$, wherc $(g^{ba})=(g_{ba})^{-1}$. Applying $F_i{}^j$ to (2.3), (i) and using (2.1) and (2.3), (i) itself, we find

$$\phi_e{}^a\phi_b{}^e=-\delta_b^a+u_bu^a,\ \ u_e\phi_b{}^e=0,\ \ \phi_e{}^au^e=0,\ \ u_eu^e=1.$$

Transvecting $F_i{}^j$ to (2.3), (ii) and taking account of (2.1) give

$$\theta_a{}^bB_b{}^j+w_aN^j=\psi_a{}^e(\phi_e{}^bB_b{}^j+u_eN^j)-v_au^bB_b{}^j,$$
$$-w^bB_b{}^j=-v^e(\phi_e{}^bB_b{}^j+u_eN^j)$$

because of (2.3), (i) and (ii). Thus we obtain

$$\phi_e{}^b\psi_a{}^e=\theta_a{}^b+v_au^b,\ \ u_e\psi_a{}^e=w_a,\ \ \phi_e{}^bv^e=w^b,\ \ u_ev^e=0.$$

Transvecting $H_i{}^j$ to (2.3), (ii) and using (2.1) imply

$$-\phi_a{}^bB_b{}^j-u_aN^j=\psi_a{}^e(\theta_e{}^bB_b{}^j+w_eN^j)-v_aw^bB_b{}^j,$$
$$-u^bB_b{}^j=-v^e(\theta_e{}^bB_b{}^j+w_eN^j)$$

because of (2.3), (i) and (iii). Thus we have

$$\theta_e{}^b\psi_a{}^e=-\phi_a{}^b+v_aw^b,\ \ w_e\psi_a{}^e=-u_a,\ \ \theta_e{}^bv^e=u^b,\ \ w_ev^e=0.$$

Similarly, using equations (2.1) and (2.3), we can prove the following formulas:

$$\phi_e{}^a\phi_b{}^e=-\delta_b^a+u_bu^a,\ \ u_e\phi_a{}^e=0,\ \ \phi_e{}^bu^e=0,\ \ u_eu^e=1,$$
$$\psi_e{}^a\psi_b{}^e=-\delta_b^a+v_bv^a,\ \ v_b\psi_a{}^b=0,\ \ \psi_e{}^bv^e=0,\ \ v_ev^e=1,$$
$$\theta_e{}^a\theta_b{}^e=-\delta_b^a+w_bw^a,\ \ w_e\theta_a{}^e=0,\ \ \theta_e{}^bw^e=0,\ \ w_ew^e=1,$$
$$\phi_e{}^b\psi_a{}^e=\theta_a{}^b+v_au^b,\ \ u_e\psi_a{}^e=w_a,\ \ \phi_e{}^bv^e=w^b,\ \ u_ev^e=0,$$
$$\theta_e{}^b\psi_a{}^e=-\phi_a{}^b+v_aw^b,\ \ w_e\psi_a{}^e=-u_a,\ \ \theta_e{}^bv^e=-u^b,\ \ w_ev^e=0,$$
$$\psi_e{}^b\theta_a{}^e=\phi_a{}^b+w_av^b,\ \ v_e\theta_a{}^e=u_a,\ \ \psi_e{}^bw^e=u^b,\ \ v_ew^e=0,$$
$$\phi_e{}^b\theta_a{}^e=-\psi_a{}^b+w_au^b,\ \ u_e\theta_a{}^e=-v_a,\ \ \phi_e{}^bw^e=-v^b,\ \ u_ew^e=0,$$
$$\theta_e{}^b\phi_a{}^e=\psi_a{}^b+u_aw^b,\ \ w_e\phi_a{}^e=v_a,\ \ \theta_e{}^bu^e=v^b,\ \ w_eu^e=0,$$

$$\psi_e{}^b\phi_a{}^e=-\theta_a{}^b+u_av^b,\quad v_e\phi_a{}^e=-w_a,\quad \psi_e{}^bu^e=-w^b,\quad v_eu^e=0.$$

Putting $\phi_{ba}=g_{ae}\phi_b{}^e$, $\psi_{ba}=g_{ae}\psi_b{}^e$ and $\theta_{ba}=g_{ae}\theta_b{}^e$, we have from (2.3)

$$\phi_{ba}=F_{ji}B_b{}^jB_a{}^i,\quad \psi_{ba}=G_{ji}B_b{}^jB_a{}^i,\quad \theta_{ba}=H_{ji}B_b{}^jB_a{}^i,$$

from which and the condition (b)

(2.5) $$\phi_{ba}=-\phi_{ab},\quad \psi_{ba}=-\psi_{ab},\quad \theta_{ba}=-\theta_{ab}.$$

We denote by ∇ the Riemannian connection induced on M from the Riemannian connection of $\overline{M}$. Then equations of Gauss and Weingarten are given by

(2.6) $$\nabla_bB_a{}^i=A_{ba}N^i,\quad \nabla_bN^i=-A_b{}^aB_a{}^i$$

respectively, A_{ba} being the components of the second fundamental tensor with respect to the unit normal vector N^i and $A_b{}^a$ being defined by $A_b{}^a=g^{ae}A_{be}$, where

$$\nabla_bB_a{}^i=\partial_bB_a{}^i+\left\{{i \atop jh}\right\}B_b{}^jB_a{}^h-\left\{{c \atop ba}\right\}B_c{}^i,$$

$$\nabla_bN^i=\partial_bN^i+\left\{{i \atop jh}\right\}B_b{}^jN^h,$$

and $\left\{{i \atop jh}\right\}$, $\left\{{c \atop ba}\right\}$ are Christoffel symbols formed respectively with g_{ji} and g_{ba}. Applying the operator $\nabla_c=B_c{}^j\overline{\nabla}_j$ to the first equation of (2.3), (i), we obtain

$$B_c{}^j(\nabla_jF_h{}^i)B_a{}^h+F_h{}^i\nabla_cB_a{}^h=(\nabla_c\phi_a{}^b)B_b{}^i+\phi_a{}^b\nabla_cB_b{}^i+(\nabla_cu_a)N^i+u_a\nabla_cN^i,$$

from which, substituting (2.2) and (2.6) and using (2.3),

$$(r_jB_c{}^j)(\psi_a{}^bB_b{}^i+v_aN^i)-(q_jB_c{}^j)(\theta_a{}^bB_b{}^i+w_aN^i)-A_{ca}u^bB_b{}^i$$
$$=(\nabla_c\phi_a{}^b)B_b{}^i+(A_{ce}\phi_a{}^e)N^i+(\nabla_cu_a)N^i-A_c{}^bu_aB_b{}^i.$$

Consequently, putting $p_c=p_jB_c{}^j$, $q_c=q_jB_c{}^j$ and $r_c=r_jB_c{}^j$, we have

$$\nabla_c\phi_a{}^b=r_c\psi_a{}^b-q_c\theta_a{}^b-A_{ca}u^b+A_c{}^bu_a,\quad \nabla_cu_a=r_cv_a-q_cw_a-A_{ce}\phi_a{}^e.$$

Similarly, using (2.2), (2.3) and (2.6), we can find

(2.7) $$\nabla_c\phi_a{}^b=r_c\psi_a{}^b-q_c\theta_a{}^b-A_{ca}u^b+A_c{}^bu_a,\quad \nabla_cu_a=r_cv_a-q_cw_a-A_{ce}\phi_a{}^e,$$

(2.8) $$\nabla_c\psi_a{}^b=-r_c\phi_a{}^b+p_c\theta_a{}^b-A_{ca}v^b+A_c{}^bv_a,\quad \nabla_cv_a=-r_cu_a+p_cw_a-A_{ce}\psi_a{}^e,$$

(2.9) $$\nabla_c\theta_a{}^b=q_c\phi_a{}^b-p_c\psi_a{}^b-A_{ca}w^b+A_c{}^bw_a,\quad \nabla_cw_a=q_cu_a-p_cv_a-A_{ce}\theta_a{}^e.$$

Let $\overline{M}$ be a $4m$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c. It is well known (Ishihara [2]) that its curvature tensor has components of the form

$$(2.10)\quad K_{kji}{}^h=\frac{c}{4}(\delta_k{}^h g_{ji}-\delta_j{}^h g_{ki}+F_k{}^h F_{ji}-F_j{}^h F_{ki}-2F_{kj}F_i{}^h+G_k{}^h G_{ji}$$
$$-G_j{}^h G_{ki}-2G_{kj}G_i{}^h+H_k{}^h H_{ji}-H_j{}^h H_{ki}-2H_{kj}H_i{}^h),$$

where c is necessary a constant, provided $m\geqq 2$.

Let M be a real hypersurface of the manifold $\overline{M}$. Then the structure equations of Gauss and Codazzi

$$K_{kjih}B_d{}^k B_c{}^j B_b{}^i B_a{}^h=K_{dcba}-A_{da}A_{cb}+A_{ca}A_{db},$$
$$K_{kjih}B_c{}^k B_b{}^j B_a{}^i N^h=\nabla_c A_{ba}-\nabla_b A_{ca}$$

are established, where $K_{kjih}=g_{ht}K_{kji}{}^t$ and $K_{dcba}=g_{ae}K_{dcb}{}^e$, $K_{dcb}{}^e$ being components of the curvature tensor determined by the induced metric g_{cb} in M. Substituting (2.3) and (2.10) into the equations above gives respectively

$$(2.11)\quad K_{dcba}=\frac{c}{4}(g_{da}g_{cb}-g_{ca}g_{db}+\phi_{da}\phi_{cb}-\phi_{ca}\phi_{db}-2\phi_{dc}\phi_{ba}$$
$$+\psi_{da}\psi_{cb}-\psi_{ca}\psi_{db}-2\psi_{dc}\psi_{ba}+\theta_{da}\theta_{cb}-\theta_{ca}\theta_{db}-2\theta_{dc}\theta_{ba})$$
$$+A_{da}A_{cb}-A_{ca}A_{db},$$

$$(2.12)\quad \nabla_c A_{ba}-\nabla_b A_{ca}=\frac{c}{4}(u_c\phi_{ba}-\phi_{ca}u_b-2\phi_{cb}u_a+v_c\psi_{ba}-\psi_{ca}v_b-2\psi_{cb}v_a$$
$$+w_c\theta_{ba}-\theta_{ca}w_b-2\theta_{cb}w_a).$$

We now denote by K_{cb} components of the Ricci tensor in M. Transvecting (2.11) with g^{da}, from (2.4) we have

$$(2.13)\quad K_{cb}=\frac{c}{4}\{(4m+7)g_{cb}-3(u_c u_b+v_c v_b+w_c w_b)\}+BA_{cb}-A_{ce}A_b{}^e,$$

where and in the sequel the mean curvature $A_e{}^e=g^{cb}A_{cb}$ will be denoted by B.

On the other hand Ishihara ([2]) proved that a $4m$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c, when $m\geqq 2$, the followings are valid:

$$\overline{\nabla}_j p_i-\overline{\nabla}_i p_j+q_j r_i-r_j q_i=-cF_{ji},$$
$$\overline{\nabla}_j q_i-\overline{\nabla}_i q_j+r_j p_i-p_j r_i=-cG_{ji},$$
$$\overline{\nabla}_j r_i-\overline{\nabla}_i r_j+p_j q_i-q_j p_i=-cH_{ji}.$$

Therefore, in the real hypersurface M, the local 1-forms p, q, r defined by

$$p_b=p_i B_b{}^i,\quad q_b=q_i B_b{}^i,\quad r_b=r_i B_b{}^i$$

satisfy

(2.14)
$$\nabla_b p_a - \nabla_a p_b + q_b r_a - r_b q_a = -c\phi_{ba},$$
$$\nabla_b q_a - \nabla_a q_b + r_b p_a - p_b r_a = -c\psi_{ba},$$
$$\nabla_b r_a - \nabla_a r_b + p_b q_a - q_b p_a = -c\theta_{ba}.$$

3. Q-Einstein hypersurfaces of a quaternionic projective space

Let M be a Q-Einstein hypersurface of a quaternionic projective space $QP(m)$. Then, by definition, the components of the Ricci tensor of M have the form

(3.1)
$$K_{cb} = \alpha g_{cb} + \beta(u_c u_b + v_c v_b + w_c w_b)$$

for some differentiable functions α and β, where u, v, w are local 1-forms defined by (2.3).

We define a symmetric tensor S of type (1, 1) by

(3.2)
$$S = A^2 - BA,$$

where A and B denote the second fundamental tensor and the mean curvature of M respectively.

LEMMA 3.1. *If M is a Q-Einstein hypersurface and $\beta \neq -3$ at every point of M, then U, V, W are eigenvectors of S whose eigenvalue is equal to $(4m+4-\alpha-\beta)$, where U, V, W are the associated vectors of u, v, w respectively. Furthermore, the other eigenvalues of S are equal to $(4m+7-\alpha)$.*

Proof. Denoting by $S_c{}^b$ the components of S and putting $S_{cb} = S_c{}^a g_{ab}$, we have

$$S_{cb} = (4m+7-\alpha)g_{cb} - (3+\beta)(u_c u_b + v_c v_b + w_c w_b)$$

because of (2.13) with $c=4$ and (3.1). This equation implies the lemma.

On the other hand, at each point we can take $X_1, \cdots, X_{4m-1}$ which are principal curvature vectors with principal curvature $r_1, \cdots, r_{4m-1}$ respectively and form as orthonormal bases. From (3.2) we get

(3.3)
$$SX_i = (r_i^2 - Br_i)X_i, \quad i = 1, \cdots, 4m-1.$$

LEMMA 3.2. *Under the assumptions of Lemma 3.1, U, V, W are principal curvature vectors.*

Proof. (3.3) means that each X_i is the eigenvector of S. Then there exist only 3 mutually orthogonal vectors X, Y, Z with eigenvalue $(4m+4-$

$\alpha-\beta)$. It follows that the eigenspace of X, Y, Z coincides with that of U, V, W. Thus we get the lemma.

We can take an orthonormal bases $U, V, W, X_4, \cdots, X_{4m-1}$ each of which is a principal curvature vector with principal curvature α_x $(x=1,2,3)$, r_i $(i=4,\cdots,4m-1)$ respectively. From Lemma 3.1 and (3.3), we have

(3.4) $$r_i^2-Br_i-(4m+7-\alpha)=0, \quad (i=4,\cdots,4m-1),$$

(3.5) $$\alpha_x^2-B\alpha_x-(4m+4-\alpha-\beta)=0, \quad (x=1,2,3).$$

Therefore, at least two of α_x's are equal, let say,

$$\alpha_1=\alpha_2.$$

Then we may assume that

(3.6) $$A_{ba}u^a=\alpha_1 u_b, \quad A_{ba}v^a=\alpha_1 v_b, \quad A_{ba}w^a=\alpha_3 w_b.$$

Now we apply the operator ∇_c to the first equation of (3.6), and take the skew-symmetric part with respect to the indices b and c. Then we have

$$(\nabla_c A_{ba}-\nabla_b A_{ca})u^a+A_{ba}\nabla_c u^a-A_{ca}\nabla_b u^a=(\nabla_c\alpha_1)u_b-(\nabla_b\alpha_1)u_c+\alpha_1(\nabla_c u_b-\nabla_b u_c),$$

from which, substituting (2.7) and (2.12) with $c=4$,

(3.7) $$2(v_c w_b-w_c v_b-\phi_{cb})+A_{ba}A_c{}^e\phi_e{}^a-A_{ca}A_b{}^e\phi_e{}^a=(\nabla_c\alpha_1)u_b-(\nabla_b\alpha_1)u_c$$
$$-(\alpha_1-\alpha_3)(q_c w_b-q_b w_c)-\alpha_1(A_{ce}\phi_b{}^e-A_{be}\phi_c{}^e).$$

By the same method we can also obtain from the second equation of (3.6)

(3.8) $$2(w_c u_b-u_c w_b-\psi_{cb})+A_{ba}A_c{}^e\psi_e{}^a-A_{ca}A_b{}^e\psi_e{}^a=(\nabla_c\alpha_1)v_b-(\nabla_b\alpha_1)v_c$$
$$+(\alpha_1-\alpha_3)(p_c w_b-p_b w_c)-\alpha_1(A_{ce}\psi_b{}^e-A_{be}\psi_c{}^e).$$

Transvecting (3.7) with u^b and using (2.4) and (3.6), we obtain

(3.9) $$\nabla_c\alpha_1=(u^b\nabla_b\alpha_1)u_c-(\alpha_1-\alpha_3)(u^b q_b)w_c.$$

Substituting (3.9) in (3.7) and transvecting with u^b, we can easily see that

$$(\alpha_1-\alpha_3)u^b q_b=0,$$

which and (3.9) imply

(3.10) $$\nabla_c\alpha_1=(u^b\nabla_b\alpha_1)u_c$$

with the help of (2.4) and (3.6).

Similarly, we can also have from (3.8) $\nabla_c\alpha_1=(v^b\nabla_b\alpha_1)v_c$, which and (3.10) give

$$\alpha_1=\text{const.} \tag{3.11}$$

Therefore, (3.7) and (3.8) reduce to respectively

$$\begin{aligned} &2(v_c w_b - w_c v_b - \phi_{cb}) + A_{ba}A_c{}^e\phi_e{}^a - A_{ca}A_b{}^e\phi_e{}^a \\ &\quad = (\alpha_3-\alpha_1)(q_c w_b - q_b w_c) - \alpha_1(A_{ce}\phi_b{}^e - A_{be}\phi_c{}^e), \end{aligned} \tag{3.12}$$

$$\begin{aligned} &2(w_c u_b - u_c w_b - \psi_{cb}) + A_{ba}A_c{}^e\psi_e{}^a - A_{ca}A_b{}^e\psi_e{}^a \\ &\quad = (\alpha_1-\alpha_3)(p_c w_b - p_b w_c) - \alpha_1(A_{ce}\psi_b{}^e - A_{be}\psi_c{}^e). \end{aligned} \tag{3.13}$$

On the other hand, from the third equation of (3.6), we can also find

$$\begin{aligned} &2(u_c v_b - u_b v_c - \theta_{cb}) + A_{ba}A_c{}^e\theta_e{}^a - A_{ca}A_b{}^e\theta_e{}^a \\ &\quad = (\nabla_c\alpha_3)w_b - (\nabla_b\alpha_3)w_c + (\alpha_3-\alpha_1)(q_c u_b - p_c v_b - q_b u_c + p_b v_c) \\ &\quad\quad -\alpha_3(A_{ce}\theta_b{}^e - A_{be}\theta_c{}^e), \end{aligned} \tag{3.14}$$

from which, transvecting w^b and using (2.4) and (3.6),

$$\nabla_c\alpha_3 = (w^b\nabla_b\alpha_3)w_c + (\alpha_3-\alpha_1)\{-(w^b q_b)u_c + (w^b p_b)v_c\},$$

which and (3.14) imply

$$(\alpha_3-\alpha_1)w^b q_b = 0, \quad (\alpha_3-\alpha_1)w^b p_b = 0. \tag{3.15}$$

Therefore,

$$\nabla_c\alpha_3 = (w^b\nabla_b\alpha_3)w_c,$$

which and (3.14) give

$$\begin{aligned} &2(u_c v_b - u_b v_c - \theta_{cb}) + A_{ba}A_c{}^e\theta_e{}^a - A_{ca}A_b{}^e\theta_e{}^a \\ &\quad = (\alpha_3-\alpha_1)(q_c u_b - p_c v_b - q_b u_c + p_b v_c) - \alpha_3(A_{ce}\theta_b{}^e - A_{be}\theta_c{}^e). \end{aligned} \tag{3.16}$$

Now we transvect w^c to (3.12) and use (2.4), (3.6) and (3.15). Then we can easily obtain

$$\alpha_1(\alpha_1-\alpha_3)v_b = (\alpha_1-\alpha_3)q_b. \tag{3.16}$$

Similarly, from (3.13) and (3.15), we also find

$$\alpha_1(\alpha_1-\alpha_3)u_b = (\alpha_1-\alpha_3)p_b. \tag{3.17}$$

Using those relations (3.16) and (3.17), we prepare the following lemma:

LEMMA 3.3. *If M is a Q-Einstein hypersurface and $\alpha+\beta\neq 4m$, $\beta\neq -3$ at every point of M, then U, V, W are principal curvature vectors with same constant principal curvature, that is,*

$$AU=\lambda U,\quad AV=\lambda V,\quad AW=\lambda W \tag{3.18}$$

for $\lambda=g(AU, U)=g(AV, V)=g(AW, W)$.

Proof. Since α_1=const. (see (3.11)), we may consider only two cases:

Case (i) $\alpha_1=0$, Case (ii) $\alpha_1\neq 0$.

In the first case (i), we have

$$\alpha_3 q_b=0,\quad \alpha_3 p_b=0$$

with the help of (3.16) and (3.17). Hence

$$q_b=0,\quad p_b=0$$

on the set $N=\{P|\alpha_3(P)\neq 0\}$. By the way the set N is void because $\phi_b{}^a=0$ on N which is a direct consequence of (2.14) with $p_b=q_b=0$. Thus we have

$$\alpha_1=\alpha_3=0.$$

In the second case (ii) we assume that $\alpha_1-\alpha_3\neq 0$. Then (3.16) and (3.17) imply

$$q_b=\alpha_1 v_b,\quad p_b=\alpha_1 u_b,$$

from which and (2.14) with $c=4$, we have

$$\alpha_1(-\alpha_1 v_c w_b+\alpha_1 v_b w_c-A_{ce}\phi_b{}^e+A_{be}\phi_c{}^e)=-4\phi_{cb}, \tag{3.19}$$

$$\alpha_1(\alpha_1 u_c w_b-\alpha_1 u_b w_c-A_{ce}\phi_b{}^e+A_{be}\phi_c{}^e)=-4\phi_{cb} \tag{3.20}$$

because of α_1=const. Transvecting (3.19) with $v^c w^b$, and using (2.4) and (3.6), we get

$$\alpha_1\alpha_3=-4,$$

which and (3.5) give

$$\alpha+\beta=4m.$$

It is contrary to our assumption and consequently

$$\alpha_1=\alpha_3$$

which implies our lemma.

From Lemma 3.3 and (3.4) we have

LEMMA 3.4. *Under the same assumptions as in Lemma* 3.3, *M has at most three distinct principal curvature at each point of M.*

On the other side, by means of Lemma 3.3, (3.12), (3.13) and (3.16) reduce to respectively

(3.21) $2(v_c w_b - w_c v_b - \phi_{cb}) - 2A_{ca}A_b{}^e\phi_e{}^a = -\lambda(A_{ce}\phi_b{}^e - A_{be}\phi_c{}^e)$,

(3.22) $2(w_c u_b - u_c w_b - \psi_{cb}) - 2A_{ca}A_b{}^e\psi_e{}^a = -\lambda(A_{ce}\psi_b{}^e - A_{be}\psi_c{}^e)$,

(3.23) $2(u_c v_b - v_c u_b - \theta_{cb}) - 2A_{ca}A_b{}^e\theta_e{}^a = -\lambda(A_{ce}\theta_b{}^e - A_{be}\theta_c{}^e)$,

where $\lambda = g(AU, U) = g(AV, V) = g(AW, W)$.

Let X be a principal curvature vector with principal curvature σ and orthogonal to U, V and W. Then, transvecting X^b to (3.21) by using $A_b{}^a X^b = \sigma X^a$, we have

(3.24) $$(2\sigma - \lambda)A_{ca}(\phi_e{}^a X^e) = (2 + \lambda\sigma)\phi_{ec}X^e.$$

Assume there exists a point P in M such that $(2\sigma - \lambda)(P) = 0$. Then $(2 + \lambda\sigma)(P) = 0$ and consequently $(2\sigma^2 + 2)(P) = 0$, which is contrary. Therefore, $2\sigma - \lambda \neq 0$ at every point of M, which and (3.24) imply

$$A_a{}^c(\phi_e{}^a X^e) = ((2 + \lambda\sigma)/(2\sigma - \lambda))\phi_e{}^c X^e.$$

Similarly, by using (3.22) and (3.23), we also have

$$A_a{}^c(\psi_e{}^a X^e) = \frac{2 + \lambda\sigma}{2\sigma - \lambda}\psi_e{}^c X^e, \quad A_a{}^c(\theta_e{}^a X^e) = \frac{2 + \lambda\sigma}{2\sigma - \lambda}\theta_e{}^c X^e.$$

Thus we have proved

LEMMA 3.5. *Under the same assumptions as in Lemma* 3.3, *for any principal curvature vector X with principal curvature σ and being orthogonal to U, V, W, ϕX, ψX and θX are also principal curvature vectors with principal curvature* $2 + \lambda\sigma/2\sigma - \lambda$.

At each point, we can take orthonormal vectors $U, V, W, X_i, \phi X_i, \psi X_i, \theta X_i$ $(i = 1, 2, \cdots, m-1)$ which are principal curvature vectors. Then any tangent vector can be expressed in the following form:

$$X = xU + yV + zW + \sum_{i=1}^{m-1} x_i X_i + \sum_{i=1}^{m-1} y_i \phi X_i + \sum_{i=1}^{m-1} z_i \psi X_i + \sum_{i=1}^{m-1} s_i \theta X_i.$$

Using the above expression of X and (2.4), we can get

LEMMA 3.6. *Under the same assumptions as stated in Lemma* 3.3, *if ϕX, ψX, θX belong to the eigenspace V_σ corresponding to the principal curvature σ for any $X \in V_\sigma$, then $\{\phi, \psi, \theta\}$ commute with A.*

By means of Lemma 3.5 we find that the only possibilities are the following cases at any point P of M.

Case (i). ϕX, ψX, θX belong to V_σ for any principal curvature vector $X \in V_\sigma$.

Case (ii). There exists a principal curvature vector $X \in V_\sigma$ such that ϕX, ψX, θX do not belong to V_σ.

We assume that exists a point P of M in Case (ii). Fix the above point P of M. From Lemma 3.5 and (3.4), we obtain

$$2(r_i^2+1)-B(2r_i-\lambda)=0, \tag{3.25}$$

where r_i denotes the principal curvature of X_i.

By the equation (3.25), we easily see that only Case (i) occurs when M is minimal. Using this fact and Lemma 3.6, we have

LEMMA 3.7. *Let M be a minimal Q-Einstein hypersurface and $\alpha+\beta\neq 4m$, $\beta\neq -3$ at every point of M. Then $\{\phi, \psi, \theta\}$ commute with A.*

Next we shall prove

LEMMA 3.8. *Let M be a Q-Einstein hypersurface and $\alpha+\beta \geqq 4(m+1)$, $\beta\neq -3$ at every point of M. Then $\{\phi, \psi, \theta\}$ commute with A.*

Proof. Let r, r' be the two real roots of (3.4). We only consider the following case by Lemma 3.3 and Lemma 3.4:

For any point P of M, the tangent space T_PM at P can be written as $T_PM = V_\lambda \oplus V_r \oplus V_{r'}$ (direct sum), where $\dim V_\lambda=3$, $r\neq r'$ and $\dim V_r=s$ $(0\leqq s\leqq 4m-4)$.

From (3.5), the mean curvature B and λ have the same sign. If there exists a principal curvature vector $X\in V_r$ such that $\phi X, \psi X, \theta X$ do not belong to V_r, then by (3.25) we have $Br=2(r^2+1)+B\lambda$. Similarly we obtain the same equation for r'. We see that B, r, r' are non-zero and have the same sign. By the definition of B, we get

$$B=\operatorname{trace} A=3\lambda+B+(s-1)r+(4m-5-s)r'$$

because of $r+r'=B$. Thus we have $s=1$ and $4m-5=s$. This is a contradiction for m=integer. Therefore, V_r and $V_{r'}$ are invariant under $\{\phi, \psi, \theta\}$, which and Lemma 3.6 give our lemma.

Combinning Therem A and Lemma 3.7, we have

THEOREM 1. *Let M be a complete minimal Q-Einstein hypersurface of $QP(m)$ and $\pi:\overline{M}\to M$ the submersion which is compatible with the Hopf fibration $S^{4m+3}(1)\to QP(m)$. If $\alpha+\beta\neq 4m$, $\beta\neq -3$ at every point of M, then $M=M^Q_{p,q}(a,b)$ for some portion (p,q) of $m-1$ and some a, b such that $a^2+b^2=1$.*

Using Theorem A and Lemma 3.8, we have

THEOREM 2. *Let M be a complete Q-Einstein hypersurface of $QP(m)$ and $\pi:\overline{M}\to M$ the submersion which is compatible with the Hopf fibration $S^{4m+3}(1)$*

$\to QP(m)$. *If* $\alpha+\beta\geq 4(m+1)$ and $\beta\neq -3$ *at every point of* M, *then* $M=M^{Q}_{p,q}(a,b)$ *for some portion* (p,q) *of* $m-1$ *and some* a, b *such that* $a^2+b^2=1$.

Now we consider an Einstein hypersurface M of $QP(m)$ of which the scalar curvature R is not smaller than $4(m+1)(4m-1)$. By means of Lemma 3.8 the induced almost contact 3-structure tensors $\{\phi,\psi,\theta\}$ commute with A. Thus, as already mentioned in Theorem A, we have

$$\nabla_c A_{ba}=-u_b\phi_{ca}-u_a\phi_{cb}-v_b\psi_{ca}-v_a\psi_{cb}-w_b\theta_{ca}-w_a\theta_{cb} \tag{3.26}$$

because of $c=4$.

Differentiating (2.13) covariantly and using (2.7)～(2.9), we have

$$\begin{aligned}&-3\{(r_cv_b-q_cw_b-A_{ce}\phi_b{}^e)u_a+u_b(r_cv_a-q_cw_a-A_{ce}\phi_a{}^e)\\&+(-r_cu_b+p_cw_b-A_{ce}\psi_b{}^e)v_a+v_b(-r_cu_a+p_cw_a-A_{ce}\psi_a{}^e)\\&+(q_cu_b-p_cv_b-A_{ce}\theta_b{}^e)w_a+w_b(q_cu_a-p_cv_a-A_{ce}\theta_a{}^e)\}\\&+B\nabla_cA_{ba}-(\nabla_cA_{be})A_a{}^e-A_{be}\nabla_cA_a{}^e=0,\end{aligned}$$

from which, transvecting u^a and taking account of (2.4) and (3.26),

$$2A_{ce}\phi_b{}^e+(B+\lambda)(v_cw_b-w_cv_b)-(B-\lambda)\phi_{cb}=0, \tag{3.27}$$

where we have used

$$A_{be}u^e=\lambda u_b,\quad A_{be}v^e=\lambda v_b,\quad A_{be}w^e=\lambda w_b \text{ and } A_{ce}\phi_b{}^e+A_{be}\phi_c{}^e=0.$$

Transvecting ϕ^{cb} to (3.27) implies

$$B=\frac{2m+1}{2m-1}\lambda. \tag{3.28}$$

On the other hand, if we transvect $\phi_a{}^b$ to (3.27) and use (2.4), then

$$A_{ca}=\frac{\lambda-B}{2}g_{ca}+\frac{B+\lambda}{2}(u_cu_a+v_cv_a+w_cw_a). \tag{3.29}$$

Differentiating (3.29) covariantly and comparing the equation obtained thus with (3.26), we can easily see that

$$B^2-\lambda^2=4, \tag{3.30}$$

which and (3.28) yield

$$\frac{\lambda-B}{2}=-\frac{1}{\sqrt{2m}},\quad \frac{\lambda+B}{2}=\sqrt{2m}.$$

Hence (3.29) becomes

(3.31) $$A_{ca}=-\frac{1}{\sqrt{2m}}g_{ca}+\sqrt{2m}(u_cu_a+v_cv_a+w_cw_a).$$

Here, we consider a Riemannian submersion $\pi:\bar{M}\rightarrow M$ compatible with the Hopf fibration $\tilde{\pi}:S^{4m+3}(1)\rightarrow QP(m)$ and denote by $\bar{A}$ the second fundamental tensor of $\bar{M}$. Then $\bar{A}$ has exactly two eigenvalues $-1/\sqrt{2m}$ and $\sqrt{2m}$ of which multiplicities are $4m-1$ and 3, respectively (for detail, see [6]). Hence, if $\bar{M}$ is complete,

$$\bar{M}=\bar{M}^{Q}_{m-1,0}(1/\sqrt{2m+1},\ \sqrt{2m}/\sqrt{2m+1}).$$

Thus we have

THEOREM 3. *Let M be a complete Einstein hypersurface of $QP(m)$ and $\pi:\bar{M}\rightarrow M$ the submersion which is compatible with the Hopf fibration $S^{4m+3}(1)\rightarrow QP(m)$. If the scalar curvature $R\geqq 4(m+1)(4m-1)$, then*

$$M=M^{Q}_{m-1,0}(1/\sqrt{2m+1},\ \sqrt{2m}/\sqrt{2m+1}).$$

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