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SURJECTIVITY OF THE TRANSPOSED MAP

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Introduction.

In this paper we shall investigate certain conditions for the transposed map of a continuous linear map from a locally convex topological space to another to be surjective. Our primary concern is the case when a continous linear map is defined between two LF-spaces. Thus a continuous linear map $u:C_0^\infty(\Omega)\to C_0^\infty(\Omega)$ where $C_0^\infty(\Omega)$ is a space of test functions on an open subset Ω of R^n offers a model for our case.

The contents of this paper is divided into three sections. In the first, we shall discuss the lifting of surjective transposed maps. In the second we give an application of the first section to a continuous linear map from an LF-space to another, while in the third we shall prove a duality between a continuous linear map from an LF-space to another and its transpose when the transpose is surjective. It follows that when the LF-spaces are reflexive Schwartz spaces, surjective transposed map is an opan mapping.

The results stated in the section 1 and 2 are based on the functional analytic version of the proof of the theorem in [4]; namely, $p(D)\mathcal{D}'(\Omega)=\mathcal{D}'(\Omega)$ if Ω is a strongly $P(D)$ convex open subset of R^n and $p(D)$ is a differential polynomial acting on $\mathcal{D}'(\Omega)$, the space of distributions on Ω. The result in the section 3 essentially says that a surjective continuous linear map from $\mathcal{D}'(\Omega)$ onto itself is an open mapping.

1. Lifting of surjective transposed map

Let X and Y be two locally convex Hausdorff topological vector spaces and $u:X\to Y$ be a continuous linear map. We denote by X' (resp, Y') the dual space of X (resp. Y) and by *Spec* X (resp. *Spec* Y) the space of continuous seminorms on X (resp. Y). The transposed map of u will be denoted by ${}^tu$ (resp. u_*). Thus ${}^tu$ (resp. u_*) maps Y' (resp. *Spec* Y) into X' (resp. *Spec* X). We shall say that u is a *monomorphism* from X to Y if

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and only if u is an injective map and u: $X \to \mathrm{Im}(u) \subset Y$ is an open map. When u is injective and $u: X \to \mathrm{Im}(u) \subset Y$ is an open map with respect to the weak topologies in X and Y, we say that u is a *weak monomorphism*. When u is surjective and open, we shall say that u is an *epimorphism*.

We recall that a classical theorem in [1] states that if X, Y are locally convex topological vector spaces and X', Y' are their dual spaces, then, for a continuous linear map $u: X \to Y$, ${}^t u$ is surjective if and only if u is a weak monomorphism.

THEOREM 1.1. *(Lifting of surjective transposed maps) Let X and Y be two locally convex topological vector spaces and $u: X \to Y$ be a continuous linear map. Let X_o and Y_o be subspaces of X and Y respectively. We suppose that $u(X_o) \subset Y_o$. Let $u_o = u|_{X_o}: X_o \to Y_o$ and*

$$v: X/X_o \to Y/Y_o$$

be the continuous linear map defined canonically by u. If ${}^t u_o$ (resp. u_{o*}) *and ${}^t v$* (resp. v_*) *are surjective, then ${}^t u$* (resp. u_*) *is surjective.*

Proof. We recall first the linear isomorphisms;

$$X_o' = X'/X_o^o, \quad (X/X_o)' = X_o^o,$$

$$Spec\ X_o = Spec\ X/X_o, \quad Spec\ (X/X_o) = X_o^{\perp},$$

where X_o^o (resp. $X_o^{\perp}$) is the polar of X_o in X' (resp. $Spec\ X$) (cf. [7]). We have a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_o & \longrightarrow & X & \longrightarrow & X/X_o & \longrightarrow & 0 \\ & & \downarrow u_o & & \downarrow u & & \downarrow v & & \\ 0 & \longrightarrow & Y_o & \longrightarrow & Y & \longrightarrow & Y/Y_o & \longrightarrow & 0 \end{array}$$

which, in duality, gives rise to the following two commutative diagrams:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & X_o^o & \xrightarrow{j} & X' & \xrightarrow{q} & X'/X_o^o & \longrightarrow & 0 \\ & & \uparrow {}^t v & & \uparrow {}^t u & & \uparrow {}^t u_o & & \\ 0 & \longrightarrow & Y_o^o & \xrightarrow{i} & Y' & \xrightarrow{p} & Y'/Y_o^o & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \uparrow & & \vdots & & \uparrow & & \\ 0 & \longrightarrow & X_o^{\perp} & \longrightarrow & Spec\ X & \longrightarrow & Spec\ X/X_o^{\perp} & \longrightarrow & 0 \\ & & \uparrow v_* & & \uparrow u_* & & \uparrow u_{o*} & & \\ 0 & \longrightarrow & Y_o^{\perp} & \longrightarrow & Spec\ Y & \longrightarrow & Spec\ Y/Y_o^{\perp} & \longrightarrow & 0 \end{array}$$

We are required to show that if ${}^tv$ (resp. v_*) and ${}^tu_o$ (resp. u_{o*}) are surjective, then so is ${}^tu$ (resp. u_*).

We think of the case of ${}^tu$ first. If $x'\in X'$, there exists $z=p(y')\in Y'/Y_o^o$ such that

$${}^tu_o(z)=q(x'), \text{ i.e., } q({}^tu(y'))=q(x')$$

so that

$$q({}^tu(y')-x')=0 \text{ and } {}^tu(y')-x'\in X_o^o.$$

Thus there exists $w\in Y_o^o$ such that

$${}^tv(w)={}^tu(y')-x'$$

and

$$x'={}^tu(y')-j({}^tv(w))={}^tu(y'-i(w)).$$

Thus ${}^tu$ is also surjective.

Exactly same arguments apply for the surjectivity of u_*. This completes the proof of the theorem 1.1.

We note that, when X and Y are Hausdorff, $u:X\to Y$ is a monomorphism if and only if u_*: $Spec\,Y\to Spec\,X$ is surjective. (cf. [7]) This observation gives the following dual version of the previous theorem.

THEOREM 1.2. (*Lifting of monomorphism*) *Let X and Y be two Hausdorff locally convex topological vector spaces and $u:X\to Y$ be a continuous linear map. Let X_o* (resp. Y_o) *be a clos ed subspace of X* (resp. Y.). *We suppose that $u(X_o)\subset Y_o$ and let*

$$u_o=u|_{X_o}:X_o\to Y_o \text{ and } v:X/X_o\to Y/Y_o$$

be the continuous linear maps defined canonically by u. If both u_o and v are monomorphisms, so is u.

We shall offer another proof of the previous theorem for the future application.

Proof. The surjectivity of $u_*:Spec\ Y\to Spec\ X$ will be proved by the

Hahn-Banach type theorem for the seminorms (cf. [7]) if we prove that, for $p \in Spec\ X$ arbitrary, there is $q \in Spec\ Y$ such that for all $x \in X$,

$$p(x) \leq q\,u(x).$$

Let $p \in Spec X$. In view of our hypothesis that u_o is a monomorphism, there is $q_o \in Spec\,Y$ such that

$$(1) \qquad p(x) \leq q_o u(x) \text{ for all } x \in X_o$$

Let us set

$$p_1 = \pi_1{}^*(\sup(p, u_* q))$$

where $\pi_1{*}$ is the retraction of $Spec\ X$ onto $(X_o)^{\perp}$, i.e.,

$$(\pi_1{}^* p)(x) = \inf_{y \in X, x-y \in X_o} p(y) \text{ for } p \in Spec X,$$

where π_1 is the canonical epimorphism $X \to X/X_o$. Now π_{1*} is a surjection of $Spec\ X/X_o$ onto $(X_o)^{\perp}$. Therfore, there is $p_2 \in Spec\ (X/X_o)$ such that

$$p_1 = \pi_{1*} p_2.$$

Next, we use the fact that v is a monomorphism. Since v_* is surjective, there is $r_1 \in Spec(Y/Y_o)$ such that

$$p_2 = v_* r_1$$

Let π_2 be the canonical epimorphism $Y \to Y/Y_o$ and $\pi_{2*}: Spec(Y/Y_o) \to Spec\,Y$ be its transpose. Let us set $r = \pi_{2*} r_1$

Let ε be an arbitrary positive number. We contend that there is an integer $k \geq 0$ such that, for all $x \in X$

$$(2) \qquad p(x) \leq (1+\varepsilon) q_o(u(x)) + kr(u(x))$$

Let us suppose that this is not true. Then, for each k, we could find $x_k \in X$ such that

$$p(x_k) > 1, \quad (1+\varepsilon) q_o(u(x_k) + k\,r(u(x_k)) \leq 1.$$

In particular, we have

$$r(u(x_k)) \leq 1/k$$

But this means (cf. Remark 1.) that

$$p_1(x_k) \leq 1/k.$$

By the definition of p_1 and the definition of $\pi_1{}^*$, there is $x_k' \in X$, $x_k'' \in X_o$ such that

$$x_k = x_k' + x_k'', \quad \sup\{p(x_k'), q_o(u(x_k'))\} \leq \frac{2}{k}.$$

We derive from this that

$$|p(x_k)-p(x_k'')| \text{ and } |q_o(u(x_k))-q_o(u(x_k''))|$$

converges to 0 as $k\to\infty$. Hence, for large k,

$$p(x_k'')>1 \text{ and } q_o(u(x_k''))<1$$

which contradicts to (1) as $x_k''\in X_o$.
This completes our proof since $q=(1+\varepsilon)q_o+kr$ for some $k>0$ satisfies

$$p(x)\leq q\,u(x) \text{ for all } x\in X.$$

REMARK 1. With the previous notations, we have $p_1=u_*r$. Indeed,

$$p_1=\pi_{1*}p_2=\pi_{1*}v_*r_1=u_*\pi_{2*}r_1 \text{ and } r=\pi_{2*}r_1.$$

We recall that the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow{\scriptstyle \pi_1} & v & \downarrow{\scriptstyle \pi_2} \\ X/X_o & \longrightarrow & Y/Y_0 \end{array}$$

yields the commutative diagram

$$\begin{array}{ccc} Spec X & \xleftarrow{u_*} & Spec\ Y \\ \uparrow{\scriptstyle \pi_{1*}} & v_* & \uparrow{\scriptstyle \pi_{2*}} \\ Spec(X/X_o) & \longleftarrow & Spec(Y/Y_o). \end{array}$$

REMARK 2. We note that in the above proof we proved more than the statement of the theorem. In fact, for p, q_o as in the proof of the theorem 1.2 and for any $\varepsilon>0$, setting $q=(1+\varepsilon)q_o+kr$ for some $k>0$, we constructed a continuous seminorm q on Y such that

$$q(y)=(1+\varepsilon)q_o(y) \text{ for all } y\in Y_o,$$

$$p(x)\leq q(u(x)) \text{ for all } x\in X.$$

2. Applications to the duals of *LF*-spaces

Let E and F be LF–spaces, i.e. strict inductive limits of countably many Frechet spaces. Let $u:E\to F$ be a continuous linear map. Let $\{E_m\}$ ($m=0, 1, \cdots$), $\{F_n\}$ ($n=0, 1\cdots$) be two sequences of definitions of E and F respectively. E' and F' will be the dual of E and F and ${}^tu$: $F'\to E'$, the transpose of u as usual.

PROPOSITION 2.1. *For every* $m=0,1,\cdots$, *there is an integer* $n\geq 0$ *such that* $u(E_m)\subset F_n$.

Proof. The subspace $E_m\cap u^{-1}(F_n)$ $(n=0,1,\cdots)$ are closed and their union is equal to E_m. Therefore one of them must have an interior point, as E_m is a Baire space. Hence it must be equal to E_m. This completes the proof.

From now on, we rename the indices m so that $u(E_m)\subset F_m$ for all $m=0,1,\cdots$. We introduce the following property:

(P) *To every* $m\geq 1$ *and for every sequence* $\{x_k\}$ *in* E_m, *if the canonical images of* $u(x_k)$ *in* F_m/F_{m-1} *converges to zero, then the canonical image of* x_k *in* E_m/E_{m-1} *converges to zero.*

We note that the property (P) is exactly same as:
To every $m\geq 1$, the canonical map $v_m:E_m/E_{m-1}\to F_m/F_{m-1}$ is a monomorphism.

We also make the following assumption:

(Q) *To every* $x'\in E'$, *there is* $q_0\in Spec\ F$ *such that for all* $x\in E_0$

$$|\langle x', x\rangle|\leq q_0(u(x))$$

We note that property (Q) is equivalent to the fact that $u_o=u|E_o:E_o\to F_o$ has the surjective transpose ${}^tu_o$, or equivalently, that u_o is a monomorphism, since E_o and F_o are Frechet spaces.

THEOREM 2.1. *If* (P) *and* (Q) *both hold, then* ${}^tu$ *is surjective.*

Proof. The surjectivity will be proved if we prove that to each $x'\in E'$, there is $q\in Spec\ F$ such that, for all $x\in E$,

$$|\langle x', x\rangle|\leq q(u(x)).$$

We shall construct a seminorm q_n $(n=1,2,\cdots)$ successively on F by induction on n in the following manner. Choose a sequence $\varepsilon_n>0$ so that $\sum_{n=1}^{\infty}\varepsilon_n<\infty$. Since E_n is a closed subspace of E_{n+1} $(n=0,1,\cdots)$ we can apply the proof of the theorem 1.2 of lifting of monomorphism (cf. Remark 2) successively to construct $q_n\in Spec\ F$ such that

$$q_{n+1}(y)=(1+\varepsilon_n)q_n(y) \text{ for } y\in F_n,$$

$$|\langle x', x\rangle|\leq q_n(u(x)) \text{ for } x\in E_{n+1}.$$

It follows immediately that $q(y)=\lim q_n(y)$ exists and q is a continuous seminorm on F, since F is barrelled and

$$q(y)=\prod_{n=0}^{\infty}(1+\varepsilon_m)q_n(y) \text{ if } y\in F_n.$$

Hence we have $|\langle x', x\rangle|\leq q(u(x))$ for all $x\in E$, this completes the proof.

3. Duality

The primary aim of this section is to prove that a differential operator $P(x, D)$ with C^∞ coefficients from $\mathcal{D}'(\Omega)$ to $\mathcal{D}'(\Omega)$ is an open map if $P(x, D)$ is surjective in a generalized form. When the differential operator has constant coefficients, this fact is proved in [3].

The method used in [3] can be applied to prove the following.

THEOREM 3.1. *A hypoelliptic linear partial differential operator $P(x, D)$: $\mathcal{D}'(\Omega)\to\mathcal{D}'(\Omega)$ is an open mapping if $P(x, D)$ is surjective and ${}^tP(x, D)$ is* hypoelliptic.

Proof. It is enough to show (cf. [3]) that

(1) for each equicontinuous subset B of $C_0^\infty(\Omega)$, $({}^tP(x, D))^{-1}(B)$ is an equicontinuous subset of $C_0^\infty(\Omega)$, and

(2) the range of ${}^tP(x, D)$ is $(C_0^\infty(\Omega), \mathcal{D}'(\Omega))$-closed. That (1) is true follows from the surjectivity of $P(x, D)$. (cf. [7] 10.1 and 16.4).

To prove (2), suppose $\{f_\alpha\}$ is a net in $C_0^\infty(\Omega)$ with $\{{}^tP(x, D)f_\alpha\}$ converging to $g\in C_0^\infty(\Omega)$ with respect to $\sigma(C_0^\infty(\Omega), \mathcal{D}'(\Omega))$. Then $\{{}^tP(x, D)f_\alpha\}$ converges to g in $\varepsilon'(\Omega)$ with respect to $\sigma(\varepsilon'(\Omega), C^\infty(\Omega))$. Since $P(x, D)$: $\mathcal{D}'(\Omega)\to\mathcal{D}'(\Omega)$ is surjective and $P(x, D)$ is hypoelliptic, the map $P(x, D)$: $C^\infty(\Omega)\to C^\infty(\Omega)$ is surjective and hence open. Therefore ${}^tP(x, D):\varepsilon'(\Omega)\to\varepsilon'(\Omega)$ has a closed range with respect to $\sigma(\varepsilon'(\Omega), C^\infty(\Omega))$. Thus there exists an element $u\in\varepsilon'(\Omega)$ such that ${}^tP(x, D)u=g$. Since $g\in C_0^\infty(\Omega)$ and ${}^tP(x, D)$ is hypoelliptic, $u\in C_0^\infty(\Omega)$. Thus the range of ${}^tP(x, D)$ is $\sigma(C_0^\infty(\Omega), \mathcal{D}'(\Omega))$-closed, completing the proof.

In the sequel, we shall prove that the theorem 3.1 is a consequence of a more general version that any surjective continuous linear map from $\mathcal{D}'(\Omega)$ onto $\mathcal{D}'(\Omega)$ is an open mapping.

We recall first

DEFINION 3.1. A locally convex Hausdorff space E is said to be a *Schwartz space* if for every balanced, closed, convex neighborhood U of 0 there exists a neighborhood V of 0 such that for every $\alpha>0$ the set V can be covered by fintely many translates of αU.

We shall use the following lemmas without proof.

LEMMA 3.1. *Let E be a semi-reflexive locally convex Hausdorff space, M*

a closed subspace of E, and E' the dual of E. Then the strong topology on E'/M^o as a dual of M coincides with the quotient topology on E'/M^o induced by the the strong topology on E' (cf. [5], *p.* 272).

LEMMA 3.2. *Let E and F be LF-spaces and T be a linear map from E to F. Let $\{F_j\}$ be a sequence of Frechet spaces defining F. We assume that $T(E)\cap F_j$ is closed for all $j=1,2,\cdots$. If either E or F is a Schwartz space, then T is a monomorphism if and only if T is a weak monomorphism* (cf. [2], p. 43).

THEOREM 3.2. *Let E and F be reflexive LF spaces and E'* (resp. F') *be the dual of E* (resp. F). *Let $u: E\to F$ be a continuous linear map and ${}^tu$: $F'\to E'$ be its transpose. If either E or F is a Schwartz space, then the followings are equivalent:*

(a) *${}^tu$ is surjective,*

(b) *${}^tu$ is an epimorphism, and*

(c) *u is a monomorphism.*

Proof. It suffices to prove that (a) implies (c) since (c)$\Longrightarrow$(b)$\Longrightarrow$(a) is obvious. In fact, $u(E)$ is a closed subspace of F. Since ${}^tu: F'\to E'$ can be decomposed as a product of $\pi: F'\to F'/(u(E))^o=(u(E))'$ and $i: F'/(u(E))^o=(u(E))'\to E'$ by the lemma 3.1, and since π and i are open, it follows that ${}^tu$ is open. Since i is surjective and hence so is ${}^tu$, (c) implies (a).

The surjectivity of ${}^tu$ is equivalent to the fact that u is a weak monomorphism. Let $\{E_n\}$ and $\{F_m\}$ $(m, n=0,1,\cdots)$ be two squences of Frechet spaces defining E and F respectivly. We shall prove the following two facts are true provided that u is a weak monomorphism:

(1) For each n, there is m such that

$$u(E_n)\subset F_m, \text{ and}$$

(2) For each p, there is q such that

$$u^{-1}(F_p\cap \mathrm{Im}\, u)\subset E_q.$$

(1) is proved in the proposition 2.1. To prove (2) note that since E' and F' are barrelled, in E and F, any subset is weakly bounded if and only if it is strongly bounded. Therefore every image under u^{-1} of a bounded set in $u(E)$ is bounded. If (2) is false, we can choose a sequence $\{y_m\}$ in $F_p\cap\mathrm{Im}\, u$ such that $u^{-1}(y_m)\notin E_m$ for each m. Multiplying, if neccessary, every y_m by a number $\varepsilon_m>0$ sufficiently small, we may suppose that the sequence $\{y_m\}$ is bounded in F_p and hence in $F_p\cap\mathrm{Im}\, u$; but its image under u^{-1} is not bounded in E. This makes a contradiction.

Now let p and q be chosen to satisfy (2) above. Let us set

$$G_p = u^{-1}(F_p \cap \mathrm{Im}\ u).$$

G_p is a closed subspace of E_q and u is an injective linear map from G_p into F_p. On the other hand, note that for each q there exists m such that $E_q \subset G_m$; it follows from (1). Therefore G_p is a Frechet space and $\{G_p\}$ $(p=0, 1, 2, \cdots)$ is a sequence defining E. We have $u(G_p) \subset F_p$ and $u^{-1}(F_p \cap \mathrm{Im}\ u) = G_p$. Note that the weak topology on the subspace (here G_p or F_p) is identical to a topology induced by the weak topology in the entire space (here E or F). We conclude by this fact that u is a weak monomorphism from G_p into F_p. Since those spaces are Frechet, u is a monomorphism. From this it follows that $u(G_p) = F_p \cap \mathrm{Im}\ u$ is a closed subspace of F_p and hence of F. Since E and F are Schwartz spaces, by Lemma 3.2 it follows that u is a monomorphism. This completes proof of our theorem.

Since $C_o^\infty(\Omega)$ is a Schwartz reflexive LF-space, we have the following

COROLLARY. *Any surjective continuous linear map from $\mathcal{D}'(\Omega)$ onto $\mathcal{D}'(\Omega)$ is open.*

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