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A GAUSS MAP ON HYPERSURFACES OF SUBMANIFOLDS IN EUCLIDEAN SPACES

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In this paper we consider a hypersurface N of a submanifold $\bar{N}$ of the Euclidean space E^m. Let ξ be a unit normal vector field on $U\subset N$ in $\bar{N}$. If S^{m-1} is the hypersphere of E^m with centre the origin $(0,\cdots,0)$ and with radius 1 and if $\xi=\sum_{i=1}^{m} a^i\frac{\partial}{\partial x^i}$, where $x^1,\cdots,x^m$ is the standard coordinate system of E^m, then the Gauss map of N in $\bar{N}$ is given by $\eta: U\to S^{m-1}$; $p\to(a^1(p),\cdots,a^m(p))$. Let $\bar{\omega}$ (resp. ω) be a volume element of the spherical image of $\bar{N}$ (resp. of N) at the point $\eta(p)$ (resp. at the point p). In section 3, we look for the connection between $\bar{\omega}$ and ω in the following cases: (a). the vector field ξ is parallel in the normal bundle $N^{\perp}$, (b). N is totally geodesic in E^m, and (c). N is totally geodesic in $\bar{N}$ and ξ determines at each point an asymptotic direction of $\bar{N}$.

1. Introduction

We shall assume throughout that all manifolds, maps, vector fields, etc. $\cdots$ are differentiable of class C^∞.

Suppose that $\bar{N}$ is a $(n+1)$-dimensional submanifold of the Euclidean space E^m $(m>n+1)$ and that N is a n-dimensional submanifold (hypersurface) of $\bar{N}$. Consider in a neighborhood U of a point $p\in N$ a unit normal vector field ξ on N in $\bar{N}$. The standard Riemann connection of E^m and the Riemann connections of $\bar{N}$ and N are respectively denoted by $\bar{\bar{D}}$, $\bar{D}$ and D.

The Weingarten map L of N in $\bar{N}$ is given by

$$\bar{D}_X\xi=L(X),\quad \forall X\in N_p, \tag{1.1}$$

and $\det L$ is the Gauss curvature at the point p of the hypersurface N of $\bar{N}$. If Y and Z are vector fields of N, then we have

$$\bar{D}_Y Z=D_Y Z+V'(Y,Z),$$

where $V'(Y,Z)$ is the second fundamental form of N in $\bar{N}$. Moreover, we find

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(the metric tensor is denoted by $< , >$)

$$\bar{D}_Y Z = D_Y Z - <L(Y), Z>\xi. \tag{1.2}$$

Let U and W be vector fields of $\bar{N}$, then

$$\bar{\bar{D}}_U W = \bar{D}_U W + \bar{V}(U, W), \tag{1.3}$$

where $\bar{V}(U, W)$ is the second fundamental form of $\bar{N}$ in E^m. From (1.2) and (1.3) it follows that

$$\bar{\bar{D}}_Y Z = D_Y Z - <L(Y), Z>\xi + \bar{V}(Y, Z). \tag{1.4}$$

But, if $V(Y, Z)$ is the second fundamental form of $\bar{N}$ in E^m, then we also have

$$\bar{\bar{D}}_Y Z = D_Y Z + V(Y, Z), \tag{1.5}$$

and so, because of (1.4) and (1.5) we find that for each two vector fields Y and Z of N

$$V(Y, Z) = -<L(Y), Z>\xi + \bar{V}(Y, Z). \tag{1.6}$$

The equation of Weingarten of N in E^m, with respect to the unit normal field ξ is given by

$$\bar{\bar{D}}_X \xi = -(A_\xi(X)) + D^\perp_X \xi, \quad \forall X \in N_p, \tag{1.7}$$

where A_ξ determines a self adjoint linear map in the tangent spaces of N and $D^\perp$ is a metric connection in the normal bundle $N^\perp$. We also have

$$\bar{\bar{D}}_X \xi = \bar{D}_X \xi + \bar{V}(X, \xi)$$

or

$$\bar{\bar{D}}_X \xi = L(X) + \bar{V}(X, \xi). \tag{1.8}$$

From (1.7) and (1.8) it follows that

$$L(X) = -(A_\xi(X)) \tag{1.9}$$

and

$$D^\perp_X \xi = \bar{V}(X, \xi), \quad \forall X \in N_p. \tag{1.10}$$

Because of (1.9) we have $\det L = \pm \det A_\xi$, which means that the Gauss curvature at the point p of the hypersurface N of $\bar{N}$ is equal to $\pm K(p, \xi_p)$, where $K(p, \xi_p)$ is the Lipschitz-Killing curvature at p of N in E^m with respect to ξ_p.

Suppose that $\bar{R}$ is the curvature tensor of $\bar{N}$ and that $U_1, \cdots, U_4$ are $\bar{N}$-vector fields, then the Gauss equation of $\bar{N}$ in E^m is given by

$$<U_1, \bar{R}(U_2, U_3)U_4> = <\bar{V}(U_2, U_1),\ \bar{V}(U_3, U_4)> - <\bar{V}(U_2, U_4),\ \bar{V}(U_3, U_1)>. \tag{1.11}$$

If $X \in N_p$, then the Riemann curvature of $\bar{N}$ at p in the two-dimensional direction (X, ξ_p) is given by

$$\bar{K}(X, \xi_p) = \frac{<X, \bar{R}(X, \xi_p)\xi_p>}{<X, X>},$$

and because of (1.11),

$$<X, \bar{R}(X, \xi_p)\xi_p> = <\bar{V}(X, X),\ \bar{V}(\xi_p, \xi_p)> - <\bar{V}(X, \xi_p),\ \bar{V}(X, \xi_p)>. \tag{1.12}$$

Let Y and Z be N-vector fields, then it follows from (1.11) that $<Y, \bar{R}(Z, \xi)\xi>$ determines a 2-covariant symmetric N-tensor field. Suppose that the principal directions of this tensor field are locally (i. e. in the domain of the unit normal field ξ) given by the orthonormal base field $e_1, \cdots, e_n$ of N. Then $<(e_i)_p,\ \bar{R}((e_i)_p, \xi_p)\xi_p>$ $(i=1, \cdots, n)$ are the extremal values of the Riemann curvatures of $\bar{N}$ at p in the two-dimensional directions of N_p which contain ξ_p (or, in other words, of the Riemann curvatures $\bar{K}(X, \xi_p)$, $\forall\ X \in N_p$).

Definition. The total normal Riemann curvature of $\bar{N}$ at the point $p \in N$ is given by

$$\mathcal{K} = \prod_{i=1}^{n} <(e_i)_p,\ \bar{R}((e_i)_p,\ \xi_p)\xi_p>.$$

2. The Gauss map

Suppose that $x^1, \cdots, x^m$ is the standard coordinate system of E^m, with coordinate vector fields $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^m}$. S^{m-1} is the hypersphere of E^m with centre the origin $(0, \cdots, 0)$ and with radius 1. For the unit normal vector field ξ, with domain U, we have $\xi = \sum_{i=1}^{m} a^i \frac{\partial}{\partial x^i}$ and a^i $(i=1, \cdots, m)$ are C^∞ functions over U.

Definition. The Gauss map of N in $\bar{N}$ is given by

$$\eta : U \to S^{m-1};\quad p \to (a^1(p), \cdots, a^m(p)).$$

Let $X \in N_p$ and consider a curve $\sigma :]-a, +a[\to N$ such that $\sigma(o) = p$ and $T_{\sigma(o)} = X$. Then $\eta \circ \sigma$ is a curve on S^{m-1} and we have

$$\eta_*(X)=T_{\eta\circ\sigma(o)}=\sum_{i=1}^{m}\frac{da_o{}^i\sigma}{dt}(o)\left(\frac{\partial}{\partial x^i}\right)_{\eta(p)}$$
$$=\sum_{i=1}^{m}T_{\sigma(o)}(a^i)\left(\frac{\partial}{\partial x^i}\right)_{\eta(p)}$$

or

$$\eta_*(X)=\sum_{i=1}^{m}X(a^i)\left(\frac{\partial}{\partial x^i}\right)_{\eta(p)}. \tag{2.1}$$

We also find $\bar{D}_X\xi=\bar{D}_X\left(\sum_{i=1}^{m}a^i\frac{\partial}{\partial x^i}\right)=\sum_{i=1}^{m}X(a^i)\left(\frac{\partial}{\partial x^i}\right)_p$ and thus, because of (1.7) and (1.8),

$$-(A_\xi(X))+D_X^\perp\xi=L(X)+\bar{V}(X,\xi)=\sum_{i=1}^{m}X(a^i)\left(\frac{\partial}{\partial x^i}\right)_p. \tag{2.2}$$

3. a. The vector field ξ is parallel in the normal bundle $N^\perp$.

In this case we have $D_X^\perp\xi=0$, $\forall X\in N_p$ and $\forall p\in U$, or

$$L(X)=\sum_{i=1}^{m}X(a^i)\left(\frac{\partial}{\partial x^i}\right)_p,\quad \forall X\in N_p \text{ and } \forall p\in U. \tag{3.1}$$

The variable point with coordinates $(a^1(q),\cdots,a^m(q))$, $q\in U$ describes a submanifold S of S^{m-1} and dim $S\leqslant n$; S is the spherical image of N in the neighborhood U of the point p.

We restrict ourselves to the case $\det L\neq 0$ at the point p.

THEOREM 1. *Suppose the $\bar{\omega}$ is a volume element of the spherical image S at the point $\eta(p)$ and that ω is a volume element of N at the point p, then*

$$\eta^*(\bar{\omega})=\pm(\det L)\omega.$$

Proof. Because of (2.1) and (3.1), we know that the vectors $\eta_*(X)$ and $L(X)$, $\forall X\in N_p$ have the same components with respect to the coordinate bases $\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^m}$ at the points $\eta(p)$ and p. But $\det L\neq 0$ at p and therefore η_* is a bijection of N_p on $S_{\eta(p)}$. In this case S is n-dimensional (moreover there exists a neighborhood of p in which η is a diffeomorphism). Let $X_1,\cdots,X_n$ be an orthonormal set of eigenvectors of L at p and denote the dual forms by $\omega_1,\cdots,\omega_n$. Then $\eta_*(X_1),\cdots,\eta_*(X_n)$ form an orthogonal base of $S_{\eta(p)}$. Consider the orthonormal base

$$\eta_*(X_i)/\langle L(X_i),\ L(X_i)\rangle^{1/2},\quad i=1,\cdots,n$$

and denote the dual forms by $\bar{\omega}_1,\cdots,\bar{\omega}_n$. If ρ_i $i=1,\cdots,n$ are the eigenvalues of L at p, then

$$<L(X_i),\ L(X_i)>^{1/2}=|\rho_i| \quad i=1,\cdots,n.$$

Thus we find $\eta^*(\bar{\omega}_i)=|\rho_i|\omega_i\ i=1,\cdots,n$ and we get

$$\eta^*(\bar{\omega}_1\wedge\cdots\wedge\bar{\omega}_n)=\eta^*(\bar{\omega}_1)\wedge\cdots\wedge\eta^*(\bar{\omega}_n)=|\rho_1\cdots\rho_n|\,\omega_1\wedge\cdots\wedge\omega_n=|\det L|\,\omega_1\wedge\cdots\wedge\omega_n,$$

which has to be proved.

REMARKS.

1. In a classical way one should formulate the statement of Theorem 1 as follows: if the vector field ξ is parallel in the normal bundle $N^\perp$, then the Gauss curvature at the point p of N in $\bar{N}$ or the Lipschitz-Killing curvature $K(p,\xi_p)$ of N is equal to the ratio of volume element of the spherical image of N and the volume element of N at p.

2. If $\det L=0$ at p, then $\dim S<n$ in a neighborhood of p, or $\dim S=n$, but in this case the function $\eta^*: F^1(S_{\eta(p)})\to F^1(N_p)$ (F^1 means the vector space of 1-forms) is no more a bijection and then $\eta^*(\bar{\omega})=0$. Thus we can say that theorem 1 remaims true for $\det L=0$.

3. Suppose that $\bar{N}$ is a hypersurface of E^m, with unit normal vector field τ and with Weingarten map $\bar{L}$, then we have

$$\bar{V}(X,\xi_p)=-<\bar{L}(X),\ \xi_p>\tau_p.$$

And therefore ξ is parallel in the normal bundle $N^\perp$ iff $\bar{L}(X)\perp\xi_p$, $\forall X\in N_p$ and $\forall p\in U$, *i.e.*, ξ determines at each point $p\in U$ a principal direction of the hypersurface $\bar{N}$.

EXAMPLES

1. Consider any hypersurface N of the $(n+1)$-dimensional Euclidean space $E^{n+1}=\bar{N}\subset E^m$ $(m>n+1)$. Then the (local) normal unit vector field ξ (or $-\xi$) of N in E^{n+1} is parallel in the normal bundle $N^\perp$ and we find the well-known geometric interpretation for the Gauss curvature of a hypersurface of an Euclidean space.

2. Consider in E^m $(m>4)$ the sphere N with parametric representation

$$x^1=a\cos u\cos v,\quad x^2=a\cos u\sin v,$$
$$x^3=a\sin u,\quad x^j=0\ \ j=4,\cdots,m,\quad a>0.$$

The vector field ξ with components $\left(\frac{\cos u\cos v}{\sqrt{2}},\ \frac{\cos u\sin v}{\sqrt{2}},\ \frac{\sin u}{\sqrt{2}},\ \frac{1}{\sqrt{2}},\ 0,\cdots,0\right)$ is clearly a C^∞ normal unit vector field on N and it is parallel in the normal bundle $N^{\perp}$. Consider the manifold $\bar{N}$, represented by

$$x^1=a\ \cos u\ \cos v+\frac{k}{\sqrt{2}}\cos u\ \cos v,$$

$$x^2=a\ \cos u\ \sin v+\frac{k}{\sqrt{2}}\cos u\ \sin v,$$

$$x^3=a\ \sin u+\frac{k}{\sqrt{2}}\sin u,$$

$$x^4=\frac{k}{\sqrt{2}},$$

$$x^s=0,\quad s=5,\cdots,m \text{ and } k\in R.$$

Then N is a hypersurface of $\bar{N}$ and it is at once clear that the relative total curvature of N with respect to ξ is equal to $1/2a^2$ at each point of N, while it is easy to see that we also have $\eta^*(\omega)=\pm\omega/2a^2$.

b. N is totally geodesic in E^m

In this case we have $V(Y,Z)=0$, for each two N-vector fields. From (1.6) it follows that

$L=0$ (i.e. N is totally geodesic in $\bar{N}$) and $\bar{V}(Y,Z)=0$. Because (1.12) we have

$$\bar{K}(X,\xi_p)=-\frac{\langle\bar{V}(X,\xi_p),\ \bar{V}(X,\xi_p)\rangle}{\langle X,X\rangle},\quad \forall X\in N_p.$$

These Riemann curvatures of $\bar{N}$ are thus always negative or zero. From now on we consider only the points $p\in N$ for which the total normal Riemann curvature of $\bar{N}$ is not zero (for the case $\mathcal{K}=0$, we can make an analogous remark as in 3a.). Since zero is an extremal value for the Riemann curvatures $\bar{K}(X,\xi_p)$, $X\in N_p$ and since the function δ: $N_p\to N_p^{\perp}$; $X\to\bar{V}(X,\xi_p)$ is linear, we must suppose, if $\mathcal{K}\neq 0$, that $m\geqslant 2n+1$, otherwise δ can not be injective. Consequently we have: if $m<2n+1$, then $\mathcal{K}=0$ at each point of N.

THEOREM 2. *Suppose that $\bar{\omega}$ is a volume element of the spherical image of N at the point $\eta(p)$ and that ω is a volume element of N at the point p, then*

$$(\eta^*(\bar{\omega}))^2=(-1)^n\mathcal{K}(\omega)^2.$$

Proof. Since $L=0$, we see, because of (2.1) and (2.2) that $\eta_*(X)$ and $\bar{V}(X,\xi_p)$, $\forall X\in N_p$, have the same components with respect to the coordinate bases $\frac{\partial}{\partial x^1},\cdots,\frac{\partial}{\partial x^m}$ at the points $\eta(p)$ and p.

Whereas $\mathcal{K}\neq 0$ at the point p, η_* will be a bijection of N_p on $S_{\eta(p)}$. Suppose that $e_1, \cdots e_n$ is an orthonormal base field of N, which determines the principal direction of the 2-covariant symmetric tensor field $<Y, \bar{R}(Z,\xi)\xi> = -<\bar{V}(Y,\xi),\ \bar{V}(Z,\xi)>$. The dual forms of $(e_1)_p, \cdots, (e_n)_p$ are denoted by $\omega_1, \cdots, \omega_n$. Since

$$[-<\bar{V}((e_i)_p,\ \xi_p),\ \bar{V}((e_j)_p,\ \xi_p)>], \qquad i,j=1,\cdots,n$$

becomes a diagonal matrix, we see that $\eta_*((e_1)_p), \cdots, \eta_*((e_n)_p)$ are pairwise orthogonal. Consider the orthonormal base

$$\eta_*((e_i)_p)/<\bar{V}((e_i)_p,\xi_p),\ \bar{V}((e_i)_p,\eta_p)>^{1/2}, \quad i=1,\cdots,n$$

and denote the dual base by $\omega_1, \cdots, \omega_n$. Remark that $-<\bar{V}((e_i)_p,\ \xi_p),\ \bar{V}((e_i)_p,\xi_p)>$ are the extremal values of the Riemann curvatures of $\bar{N}$ at the point p in the two-dimensional directions of $\bar{N}_p$ which contain ξ_p. We have

$$\eta^*(\bar{\omega}_i) = <\bar{V}((e_i)_p,\xi_p),\ (\bar{V}((e_i)_p,\xi_p)>^{1/2}\omega_i, \qquad i=1,\cdots,n$$

and so we find

$$\begin{aligned}\eta^*(\bar{\omega}_1\wedge\cdots\wedge\bar{\omega}_n) &= \eta^*(\bar{\omega}_1)\wedge\cdots\wedge\eta^*(\bar{\omega}_n)\\ &= \prod_{=1}^{n}<\bar{V}((e_i)_p,\xi_p),\ \bar{V}((e_i)_p,\xi_p)>^{1/2}\ \omega_1\wedge\cdots\wedge\omega_n\\ &= \sqrt{(-1)^n\mathcal{K}}\,\omega_1\wedge\cdots\wedge\omega_n,\end{aligned}$$

and this completes the proof.

EXAMPLE. A variable n-dimensional linear space $N(s)$ which is dependent on one parameter s, describes a monosystem $\bar{N}$ in E^m. If $r(s)$ is a base curve and if $a_1(s), \cdots, a_n(s)$ constitutes a base of the variable generating space $N(s)$, then $\bar{N}$ can (locally) be represented by

$$X(s, 1_1, \cdots, 1_n) = r(s) + \sum_{i=1}^{n} 1_i a_i(s), \quad 1_i\in R,\ i=1,\cdots,n.$$

Each generating space $N(s)$ is a hypersurface of $\bar{N}$, which is totally geodesic in E^m. If (accents mean derivation to s)

$$\text{rank}\ [r'(s)\ \ a_1(s)\cdots a_n(s)\ \ a_1'(s)\cdots a_n'(s)] = 2n+1,\ \forall s,$$

then $\bar{N}$ is non-developable. In this case it can be proved that at each point of each generating space $N(s)$ we have $\mathcal{K}\neq 0$ and so we can apply Theorem 2 (see [4]).

c. N is totally geodesic in $\bar{N}$ and $\bar{V}(\xi_p,\xi_p)=0$, $\forall p\in N$.

In this case the second fundamental form of N in $\bar{N}$ is identically zero, *i.e.* $L=0$ at each point p of N. If $\overline{V}(\xi_p, \xi_p)=0$, $\forall p\in N$, then the vector ξ_p determines at each point $p\in N$ an asymptotic direction of $\overline{N}$. Because of (1.12), we find

$$K(X,\xi_p)=-\frac{<\overline{V}(X,\xi_p),\overline{V}(X,\xi_p)>}{<X,X>},\quad \forall X\in N_p.$$

These Riemann curvatures of N are always negative or zero. We consider only the points $p\in N$, for which the total normal Riemann curvature $\mathscr{K}$ of $\overline{N}$ is not zero and therefore we must suppose, analogously as in 3.b, that $m\geqslant 2n+1$ (if $m<2n+1$, then we have again $\mathscr{K}=0$ a each point of N).

THEOREM 3. *Suppose that* $\bar{\omega}$ *is a volume element of the spherical image* S *of* N *at the point* $\eta(p)$ *and that* ω *is a volume element of* N *at the point* p, *then*

$$(\eta^*(\bar{\omega}))^2=(-1)^n\mathscr{K}(\omega)^2. \tag{3.2}$$

Proof. The proof of this statement is totally analogous to that of Theorem 2.

EXAMPLE. Suppose that $x^1,\cdots,x^{2(n+1)}$ are orthonormal coordinates in $E^{2(n+1)}$ and consider E^{n+1} as the subspace of $E^{2(n+1)}$ represented by $x^{n+2}=\cdots=x^{2(n+1)}=0$. In E^{n+1} we take a hypersurface N, which is locally given by the following parametric representation (u_j $j=1,\cdots,n$ are the parameters)

$$\begin{aligned} &x^1=f_i(u_1,\cdots u_n),\quad i=1,\cdots,n+1,\\ &x^k=0,\quad k=n+2,\cdots,2(n+1). \end{aligned} \tag{3.3}$$

Using the unit normal vector field $\tau(\tau_1(u_1,\cdots,u_n),\cdots,\tau_{n+1}(u_1,\cdots,u_n),0,\cdots,0.)$ of N in E^{n+1}, we construct the following $(n+1)$-dimensional submanifold N of $E^{2(n+1)}$:

$$\begin{aligned} &x^1=f_i(u_1,\cdots,u_n)\quad i=1,\cdots,n+1,\\ &x^j=1\tau_{j-n-1}(u_1,\cdots,u_n)\quad j=n+2,\cdots,2(n+1) \text{ and } 1\in R. \end{aligned}$$

Then N (or the part N given by (3.3), which will henceforth be denoted by N) clearly is a hypersurface of $\overline{N}$.

Consider in $E^{2(n+1)}$ the normal unit vector field ξ on N, with components $(0,\cdots,0,\ \tau_1(u_1,\cdots,u_n),\cdots,\tau_{n+1}(u_1,\cdots,u_n))$. If $\bar{\bar{D}}$ is the standard Riemann connection of $E^{2(n+1)}$, $\overline{D}$ the Riemann connection of $\overline{N}$, $\overline{V}$ the second fundamental form of $\overline{N}$, L the Weingarten map of N in $\overline{N}$ and if $\overline{L}$ is the Weingarten map of N in E^{n+1}, then we have

$$\bar{\bar{D}}_X\tau=\overline{L}(X),\quad \forall X\in N_p \tag{3.4}$$

and the Gauss curvature G of N in E^{n+1} at the point p is given by $\det\overline{L}$.

Moreover we find

$$\bar{\bar{D}}_X\xi=\bar{D}_X\xi+\bar{V}(X,\xi)=L(X)+\bar{V}(X,\xi). \quad (3.5)$$

But if we consider the components of the unit normal field ξ, then it is clear that $L=0$ at each point of N, i.e., N is totally geodesic in $\bar{N}$. We also have that ξ determines at each point of N an asymptotic direction of $\bar{N}$, i.e., $\bar{V}(\xi,\xi)=0$. It is also clear that

$$<\bar{\bar{D}}_X\tau,\ \bar{\bar{D}}_X\tau>=<\bar{\bar{D}}_X\xi,\ \bar{\bar{D}}_X\xi>. \quad (3.6)$$

Because of (3.4), (3.5) and (3.6) we find for the Riemann curvature of $\bar{N}$ in the two-dimensional direction (X,ξ_p) of $\bar{N}_p$

$$\bar{K}(X,\xi_p)=-\frac{<\bar{V}(X,\xi_p),\ \bar{V}(X,\xi_p)>}{<X,X>}=-\frac{<\bar{L}(X),\bar{L}(X)>}{<X,X>}.$$

If the principal curvatures of the hypersurface N (of E^{n+1}) at the point p are denoted by $1/R_i$, $i=1,\cdots,n$, then we have at once for total normal Riemann curvature of $\bar{N}$ at p

$$\mathcal{K}=\prod_{i=1}^{n}(-1)^n\frac{1}{R_i^2}=(-1)^n\ G^2,$$

and this is what (3.2) says, because in our example $\eta^*(\omega)=\pm|G|\omega$.

REMARK

1. If $n=1$, than N is a curve on the surface $\bar{N}$. Suppose that T is a unit tangent vector field of N and that the unit normal vector field ξ is parallel in the normal bundle $N^\perp$, then we have $\bar{D}_T\xi=kT=L(T)$ for some $k\in R$ and Theorem 1 remains true ($k=\det L$ and volume element is now arc element). This is also valid for Theorems 2 and 3.

Remark that in the case $n=1$, the total normal Riemann curvature $\mathcal{K}$ of $\bar{N}$ at the point p of N is equal to the Riemann curvature (or Gauss curvature) G of $\bar{N}$ at p. We give an example for the third case (3.c): consider a non-developable ruled surface $\bar{N}$ in E^n $(n\geqslant 3)$, which is locally represented by

$$r(s)+1\xi(s),\ \ \xi^2=1,\ \ s\in I\subset R,\ \ 1\in R,$$

where s is the arc length of the base curve $I\to\bar{N}$; $s\to r(s)$, which is an orthogonal trajectory of the generating lines. Suppose that $r(s)$ (which is in this example N) is a geodesic of $\bar{N}$. Then, a theorem of Bonnet says that N is also the line of striction of $\bar{N}$ and the conditions for all this are (with classical notations; accents mean derivation to s) $r'\xi=r'\xi'=0$, $\forall s\in I$. In this case the parameters of distribution d are given by $d^2=1/\xi'^2$, $\forall s\in I$ and for the Riemann curvature K of $\bar{N}$ at q we find $G=-d^2/(d^2+t^2)^2$, where t is the distance between q and the point of striction on the generating line through q. At the points p of N we have $t=0$ and thus

$$G=-\frac{1}{d^2}=-\xi'^2=-\frac{(d\xi)^2}{(ds)^2},$$

and this is what (3.2) says.

2. Suppose that N is totally geodesic in E^m or that N is totally geodesic in $\bar{N}$ and $\bar{V}(\xi_q,\xi_q)=0$, $\forall q\in N$. Take a point $p\in N$ and a vector $X\in N_p$. Consider a curve $\sigma:]-a,+a[\to N;\ t\to\sigma(t)$ on N, such that $\sigma(o)=p$ and $T_{\sigma(0)}=X$. Then we have for the arc length s of σ

$$\left(\frac{ds}{dt}\right)_{t=0}=\langle X,X\rangle^{1/2}. \tag{3.7}$$

The spherical image of σ is the curve $\eta\circ\sigma$ on S. We find for the arc length $\bar{s}$ of the curve $\eta\circ\sigma$, because of (2.1) and (2.2),

$$\begin{aligned}\left(\frac{d\bar{s}}{dt}\right)_{t=o}&=\langle T_{\eta\circ\sigma(o)},\ T_{\eta\circ\sigma(o)}\rangle^{1/2}\\ &=\langle\sum_{i=1}^{m}X(a^i)\left(\frac{\partial}{\partial x^i}\right)_{\eta(p)},\ \sum_{i=1}^{m}X(a^i)\left(\frac{\partial}{\partial x^i}\right)_{\eta(p)}\rangle\\ &=\langle\bar{V}(X,\xi_p),\ \bar{V}(X,\xi_p)\rangle.\end{aligned} \tag{3.8}$$

Now the expressions of $\bar{K}(X,\xi_p)$ in the cases 3.b. and 3.c., together with (3.7) and (3.8) give a nice geometrical interpretation of such Riemann curvature of $\bar{N}$: suppose that $t=o$ gives $s=o$)

$$\bar{K}(X,\xi_p)=-\left(\frac{d\bar{s}}{ds}\right)^2_{s=0}.$$

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