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NOTE ON THE SEMIGROUP OF FUZZY MATRICES

BY JIN BAI KIM

1. Introduction

Let F be a finite subset of the unit interval $[0, 1]$ of the real line. $M_n(F)$ denotes the set of all $n\times n$ fuzzy relation matrices over F. Then $M_n(F)$ form a semigroup under the matrix multiplication and we call it the semigroup of fuzzy (relation) matrices over F [5]. We denote by $C_n(F)$ the set of all circulant fuzzy (relation) matrices over F. Then $C_n(F)$ is a commutative subsemigrop of $M_n(F)$ [6]. [5, 6, 7] are main references for $M_n(F)$ and $C_n(F)$. We define $kI_r=(a_{ij})$ by $a_{ij}=k$ if $i=j=1, 2, \cdots, r$ and $a_{ij}=0$ if $i\neq j$ or if $i=j=r+1,\ r+2, \cdots, n$. Let $R^n{}_{rk}$ and $D^n{}_{rk}$ be respectively an R-class and a D-class containing $kI_r\in M_n(F)$, where k is a member of F. We find the cardinal numbers $|R^n{}_{rk}|$ and $|D^n{}_{rk}|$ of the sets $R^n{}_{rk}$ and $D^n{}_{rk}$. We find the number of all idempotents in $D^n{}_{rk}$.

2. Elementary properties of the semigroup

We list some elementary properties of $M_n(F)$ and $C_n(F)$.

(1) $M_n(F)$ is not regular. $C_n(F)$ is an abelian subsemigroup of $M_n(F)$ with the cardinality $|C_n(F)|=(|F|)^n$.

(2) If $A\in C_n(F)$, then A^h is an idempotent for some $h\leqq n$.

(3) Any idempotent A in $C_n(F)$ is of the form

$$A=a_1E+a_2(P^{d_1}+P^{2d_1}+\cdots+P^{(t_1-1)d_1})+\cdots+a_m(P^{d_m}+P^{2d_m}+\cdots+P^{(t_m-1)d_m}),$$

where $n=d_it_i$, $d_1=d_2s_2$, $d_2=d_3s_3, \cdots$, $d_{m-1}=d_ms_m$, $d_1\neq n$, $d_i\neq d_j$ $(i\neq j$, $a_i\in F$, $a_1\geq a_2>a_3>\cdots>a_m\neq 0$, and E is the identity of $M_n(F)$. (See [5, 6] for proofs of items (2) and (3).

(4) Let $r(A)$ and $c(A)$ denote respectively the row-rank and column-rank of $A\in M_n(F)$. Then $r(A)=1$ iff $c(A)=1$. If $D_1=\{X\in M_n(F):\ \mathrm{r}(X)=1\}$, then D_1 is the union of k D-classes of $M_n(F)$, where $k=|F|-1$. For simplicity, we write $F=\{0, 1, 2, \cdots, k\}$.

We denote by $D^n{}_{ra}$ the D-class of $M_n(F)$ containing aI_r, $0\neq a\in F$.

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THEOREM 1. *Let* $0\neq a\neq b\neq 0$. *Then* $D^n{}_{ra}\cap D^n{}_{rb}=\phi$ (*the empty set*). *Let* $D^n{}_r$ *be the union of all* $D^n{}_{ra}$ $(0\neq a\in F)$. *If* $A\in D^n{}_r$, *then* $r(A)=c(A)$.

Proof. Assume that $a>b$. There is no X in $M_n(F)$ such that $aI_r=(bI_r)X$ and consequently we have shown that $D^n{}_{ra}\cap D^n{}_{rb}=\phi$. We show that if $A\in D^n{}_r$ then $r(A)=r=c(A)$. To show that let $A\in D^n{}_{ra}$. Then there exists B such that $A\,L\,B$ and $B\,R\,(aI_r)$. There exist X, Y, U, V such that $BX=aI_r$, $UA=B$, $aI_rY=B$ and $VB=A$. We obtain $C(aI_r)\subseteq C(B)\subseteq C(aI_r)$ and hence $C(aI_r)=C(B)$, where $C(B)$ denotes the column space of B. Thus $c(B)=c(aI_r)=r(aI_r)=r$. Since B has at most r non-zero rows, we have that $r(B)\leqq r$. From $c(B)=r$, it follows that there exists a submatrix $G=(c_{ij})$ of B of order r such that $c_{ii}=a$ $(i=1,2,\cdots,r)$ and $c_{ij}=0$ for $i\neq j$. This means that $r(B)=r$. Now from $UA=B$ and $VB=A$, we have that $R(A)=R(VB)\subseteq R(B)=R(UA)\subseteq R(A)$ and $R(A)=R(B)$, where $R(B)$ denotes the row space of B. We obtain $r(A)=r(B)=r$. Similarly, we can show that $c(A)=c(B)=r$. This proves Therem 1.

3. The R-class $R^n{}_{rm}$

We note that $F=\{0,1,2,\cdots,k\}$. Let $0\neq m\in F$. $R^n{}_{rm}$ denotes the R-class containing mI_r. We compute the cardinality $|R^n{}_{rm}|$ of the set $R^n{}_{rm}$ in the following theorem. $\binom{n}{k}$ denotes the binomial coefficient.

THEOREM 2. *Let r be a positive integer with* $1\leqq r\leqq n$. *Then*

$$|R^n{}_{rm}|=\sum_{i=0}^{r}(-1)^i\binom{r}{i}((m+1)^r-i)^n.$$

To prove the theorem 2 we need lemmas.

LEMMA 1. $\binom{t}{m}+\binom{t}{m+1}=\binom{t+1}{m+1}$.

LEMMA 2. $|R^n{}_{1m}|=(m+1)^n-m^n$.

Proof. We supply two proofs of Lemma 2. $V_n(m)$ denotes the set of all $1\times n$ matrices over $F=\{0,1,2,\cdots,m\}$. Let $\underline{0}$ denote the $(n-1)\times n$ zero matrix. Letting $x=(x_1,x_2,\cdots,x_n)\in V_n(m)$, $\underline{x}=\binom{x}{0}$ denotes an $n\times n$ matrix formed from x and $\underline{0}$. Then $\underline{x}\in R^n{}_{1m}$ iff x contains at least one m as its component. We see that $|V_n(m)|=(m+1)^n$. Therefore we see that

$$|R^n{}_{1m}|=|V_n(m)|-|V_n(m-1)|=(m+1)^n-m^n,$$

proving the lemma.

The second proof is given by the following expression.

$$|R^n{}_{1m}| = \binom{n}{1}\left(\sum_{t=0}^{n-1}\binom{n-1}{t}(m-1)^t\right) + \binom{n}{2}\left(\sum_{t=0}^{n-2}\binom{n-2}{t}(m-1)^t\right) + \cdots$$
$$+ \binom{n}{i}\left(\sum_{t=0}^{n-i}\binom{n-i}{t}(m-1)^t\right) + \cdots + \binom{n}{n} = (m+1)^n - m^n.$$

A term of $|R^n{}_{1m}|$ with the coefficient $\binom{n}{i}$ expresses the number of $\underline{x} = \binom{x}{0}$ of $R^n{}_{1m}$ such that $|\{j\colon x_j = m\}| = i$, where $x = (x_1, x_2, \cdots, x_n) \in V_n(m)$. This also proves the lemma.

LEMMA 3. *Let* $F = \{0, 1, 2, \cdots, k\}$. *Let* $V_r(F)$ *denote the row vector space over* F. *Then* $|V_r(F)| = (k+1)^r$.

Proof of Theorem 2. We prove this theorem by induction on r. We proved this theorem for case $r=1$ in Lemma 2. We assume that Theorem 2 is proved for all r less than r_0, where r_0 being a fixed positive integer less than n and greater than 1.

Consider $|R^n{}_{r_0 m}|$. By induction assumption we have that

$$|R^n{}_{r_0-1\, m}| = \sum_{i=0}^{r_0-1}(-1)^i\binom{r_0-1}{i}((m+1)^{r_0-1}-i)^n = Q.$$

The first term of Q is $((m+1)^{r_0-1})^n$ which becomes

$$((m+1)^{r_0})^n - ((m+1)^{r_0}-1)^n = (u)^n - (u-1)^n$$

if we assume $R^n{}_{r_0-1\, m}$ has changed to $R^n{}_{r_0\, m}$, by a careful application of Lemma 4, where $u = (m+1)^{r_0}$ which is justified by Lemma 3. Consider the second term $-(r_0-1)((m+1)^{r_0-1}-1)^n$ of Q. This quantity becomes $-(r_0-1)(v^n-(v-1)^n)$ as $R^n{}_{r_0-1\, m}$ becomes $R^n{}_{r_0 m}$ by Lemma 4, where $v = (m+1)^{r_0}-1$ which is justified by Lemma 3. In similar fashin, by applying Lemma 4 and Lemma 3 to each term $(-1)^i\binom{r_0-1}{i}((m+1)^{r_0-1}-i)^n = q$ of Q, we claim that q becomes

$$(-1)\binom{r_0-1}{i}(((m+1)^{r_0}-i)^n - ((m+1)^{r_0}-(i+1))^n)$$

as $R^n{}_{r_0-1\, m}$ becomes $R^n{}_{r_0 m}$. Now using the identity $\binom{t}{s-1}+\binom{t}{s}=\binom{t+1}{s}$ of Lemma 1, we obtain

$$|R^n{}_{r_0 m}| = ((m+1)^{r_0})^n - \binom{r_0}{1}((m+1)^{r_0}-1)^n + \cdots + (-1)^{r_0}\binom{r_0}{r_0}((m+1)^{r_0}-r_0)^n$$
$$= \sum_{i=0}^{r_0}(-1)^i\binom{r_0}{i}((m+1)^{r_0}-i)^n.$$

This proves Theorem 2.

THEOREM 3. $|D^n{}_{rm}|=\frac{1}{r!}(|R^n{}_{rm}|)^2$.

Proof. We can prove that $|L^n{}_{rm}|=|R^n{}_{rm}|$, where $L^n{}_{rm}$ denotes the L-class containing mI_r. We can also show thow that $|H^n{}_{rm}|=r!$, where $H^n{}_{rm}$ denotes the H-class containing mI_r. This proves Theorem 3.

4. The number of all idempotents in $D^n{}_{rm}$

$E(S)$ denotes the set of all idempotents of a subset S of a semigroup. We find $|E(D^n{}_{rm})|$ in Theorem 4.

LEMMA 4. $|E(D^n{}_{1m})|=((m+1)^2)^n-((m+1)^2-1)^n$.

We give two different proofs of Lemma 4.

Proof. (1). Let $x=(x_1,x_2,\cdots,x_n)\in V_n(F)$ and $\underset{\sim}{0}$ denotes the $(n-1)\times n$ zero matrix. Then $\binom{x}{\underset{\sim}{0}}\in R^n{}_{1m}$ if $x_i=m$ for some i. $\underset{\sim}{x}=\binom{x}{\underset{\sim}{0}}\in R^n{}_{1m}$ is an idempotent iff $x_1=m$. Thus the number of all idempotents in $R^n{}_{1m}$ is equal to $(m+1)^{n-1}$. It is not difficult to show that there are u R-classes each of which contains exactly $(m+1)^{n-1}$ idempotents, where $u=\binom{n}{1}m^{n-1}$. Now consider an R-class R_A which contains $A=(a_{ij})$, where $a_{11}=a_{21}=m$ and $a_{ij}=0$ if $a_{11}\neq a_{ij}\neq a_{21}$. We can show that $R_A\subseteq D^n{}_{1m}$ and $|E(R_A)|=(m+1)^{n-2}((m+1)^2-m^2)$. We can show that there are exactly u R-classes R_B such that $|E(R_A)|=|E(R_B)|$ and $R_B\subset D^n{}_{1m}$, where $u=\binom{n}{2}m^{n-2}$. By the foregoing argument, we can write

$$\begin{aligned}|E(D^n{}_{1m})|&=\binom{n}{1}m^{n-1}(m+1)^{n-1}+\binom{n}{2}m^{n-2}(m+1)^{n-2}((m+1)^2-m^2)+\\&\quad\cdots+\binom{n}{r}m^{n-r}(m+1)^{n-r}((m+1)^r-m^r)+\cdots+\binom{n}{n}((m+1)^n-m^n)\\&=(m+1)^{2n}-((m+1)^2-1)^n.\end{aligned}$$

This proves Lemma 4.

(2) (The second proof of Lemma 4). For simplicity we write $F=\{0,1,2,\cdots,m\}$ instead of $F=\{0=r_1,r_2,\cdots,r_m=1:r_i\in[0,1]\}$. Consider $E(D^n{}_{1m})$. $A\in E(D^n{}_{1m})$ iff there exist $x=(x_1,x_2,\cdots,x_n)^t$ (t means that 'transpose') and $y=(y_1,y_2,\cdots,y_n)$ such that $xy=A$ and $yx=m$. This shows that if $A\in$

$E(D^n{}_{1m})$ then there exists i such that $x_i=m=y_i$ and $xy=A$. Now we consider the meaning of $(m+1)^{2n}-m^{2n}$. $(m+1)^{2n}$ means the number of $xy=A$ such that $x=(x_1, x_2, \cdots, x_n)^t$ and $y=(y_1, y_2, \cdots, y_n)$, $x_i, y_i \in F$. m^{2n} means that

$$|\{xy : x_i, y_j \in \{0, 1, 2, \cdots, m-1\} = F\backslash m\}| = m^{2n}.$$

This proves that $|E(D^n{}_{1m})| = (m+1)^{2n} - m^{2n}$.

Note that if we set $r=1$, in Theorem 4, then $2(m+1)^r + rm^2 - 1 = (m+1)^2$. If $r=2$, then $2(m+1)^r + rm^2 - 1 = (2m+1)^2$.

DEFINITION. We denote by $M_{m,n}(F)$ the set of all $m\times n$ matrices over F. We define a square matrix $I(m,r)=(a_{ij})$ of order r as the following: $a_{ii}=m$ and $a_{ij}=0$ for all $i\neq j$. We define

$$X=\{\underline{x}=\begin{pmatrix} I(m,r) \\ x \end{pmatrix} \in M_{r+1,r}(F)\} \text{ and } Y=\{\underline{y}=(I(m,r), y) \in M_{r,r+1}(F)\},$$

where $x=(x_1, x_2, \cdots, x_r)$ and $y=(y_1, y_2, \cdots, y_r)^t$. Note that $\underline{x}\underline{y}=$is a member of $M_{r+1,r+1}(F)$ for $\underline{x}\in X$ and $\underline{y}\in Y$.

LEMMA 5. *Let* $\underline{x}Y=\{\underline{x}\underline{y} : \underline{x}\in X, \ \underline{y}\in Y\}$. *Then*

(1) $|E(\underline{x}Y)|=(m+1)^r$ *when* $x=(0, 0, \cdots, 0)$.

(2) $|E(\underline{x}Y|=m+1$ *when* $x=(x_1, x_2, \cdots, x_r)$ *contains just one nonzero element.*

(3) $|E(\underline{x}Y)|=1$ *when* x *contains at least two nonzero components.*

The proof of Lemma 5 is trivial and we omit it.

Lemma 5 has an important meaning in Theorem 4. For (2) of Lemma 5, there exist mr such vectors $x=(x_1, x_2, \cdots, x_r)$ each of which contains just one non-zero element (referring to $F=\{0, 1, 2, \cdots, m\}$). We define a number $rm(m+1)$ for (2). Consider (1) of Lemma 5. There is just one vector $x=(0, 0, \cdots, 0)$, the zero-vector, and we define a number $(m+1)^r$ for (1). For (3), we define a number $(m+1)^r-mr-1$. The sum of these three numbers $rm(m+1)$, $(m+1)^r$ and $(m+1)^r-mr-1$ is equal to $2(m+1)^r+rm^2-1=t$ and t has a significant meaning for $|E(D^n{}_{rm})|$.

THEOREM 4. $$|E(D^n{}_{rm})| = \frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(t-i)^n,$$
where $t=2(m+1)^r+rm^2-1$.

Proof. We prove this theorem by induction on r. For the case of $r=1$, we have Lemma 4 which proves Theorem 4. We assume that the theorem is proved for all $r<r_0$, where r_0 is a fixed positive integer less than n. This means that

$$|E(D^n{}_{r_0-1\,m})|=\frac{1}{(r_0-1)!}\sum_{i=0}^{r_0-1}(-1)^i\binom{r_0-1}{i}(t(r_0-1)-i)^n,$$

where $t(s)=2(m+1)^s+sm^2-1$. Consider A in $D^n{}_{r_0m}$. For A there exist $U\in L^n{}_{r_0m}$ and $V\in R^n{}_{r_0m}$ such that $A=UV$. For a moment, we suppose that A were $\bar{A}=\bar{U}\bar{V}$, $\bar{U}\in L^n{}_{r_0-1\,m}$, $\bar{V}\in R^n{}_{r_0-1\,m}$. Applying Lemma 4 to $\bar{A}$ which changes to A, and a careful consideration of the meaning of Lemma 5, we can realize that the first term $\binom{r_0-1}{0}(t(r_0-1))^n$ of $|E(D^n{}_{r_0-1\,m})|$ changes to $\binom{r_0-1}{0}(t(r_0))^n-(t(r_0)-1)^n)$ as $\bar{A}$ changes to A or r_0-1 changes to r_0. The foregoing argument applies to the second term $\binom{r_0-1}{1}(t(r_0-1)-1)^n$ which changes to $\binom{r_0-1}{1}((t(r_0)-1)^n-(t(r_0)-2)^n)$ as $\bar{A}$ changes to A. We apply the foregoing argument to each term of $|E(D^n{}_{r_0-1m})|$ and using $\binom{s}{m}+\binom{s}{m-1}=\binom{s+1}{m}$ of Lemma 1, we obtain the formula for $|E(D^n{}_{r_0m})|$. This proves the theorem.

5. Some additional results

It is difficult to find the number of all D-classes of the semigroup $M_n(F)$ but we have the following.

PROPOSITION 1. *The number of all D-classes of the semigroup $M_2(F)$ of all 2×2 fuzzy matrices over F is given by* $\sum_{t=0}^{m}(2t^2-t+1)(m-t+1)$, *where m is given by $|F|=m+1$.*

Note that if $m=1$, then the number above is equal to 4. We know that the semigroup $M_2(\{0,1\})$ of all 2×2 boolean matrices over the set $\{0,1\}$ of two elements has four D-classes:

$$D_{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)},\quad D_{\left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)},\quad D_{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)}\ \text{and}\ D_{\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)}.$$

We consider now Theorem 2. Let $A\in M_n(F)$. $R(A)$ and $L(A)$ denote respectively the row space and the column space of A. R_A and L_A denote respectively the R-class and the L-class containing A. The following is the generalized from of Theorem 2.

PROPOSITION 2. $|R_A|=\sum_{i=0}^{c(A)}(-1)^i\binom{c(A)}{i}(|L(A)|-i)^n$ *and*

$$|L_A|=\sum_{i=0}^{r(A)}(-1)^i\binom{r(A)}{i}(|R(A)|-i)^n.$$

The proof of Proposition 2 is similar to that of Theorem 2 and we omit the proofs of Propositions 1 and 2.

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West Virginia University, U. S. A.