J. Korean Math. Soc.
Vol. 15, No. 2. 1979

ABSOLUTE MATRIX SUMMABILITY FACTORS OF INFINITE SERIES AND THEIR APPLICATION TO FOURIER SERIES

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1. Definitions and notations

Let s_n be the n-th partial sum of a given infinite series $\sum_{n=0}^{\infty} a_n$ and let $B=(b_{n,m})$ be a normal matrix (see [11]). Then we define the sequence to sequence transformation of a sequence (s_n) by

(1.1) $$t_n=B(s_n)=\sum_{m=0}^{\infty} b_{n,m}s_m.$$

We say that $\sum_{n=0}^{\infty} a_n \in |B|$, if

(1.2) $$\sum_{n=0}^{\infty} |t_n-t_{n-1}|<\infty,$$

where, by convention, we assume $t_{-1}=0$.

Let (p_n) be a sequence of constants such that $P_n=p_0+p_1+\cdots+p_n\neq 0$ and $P_{-n}=p_{-n}=0$ $(n\geqslant 1)$, and let

(1,3) $$b_{n,m}=\begin{cases} p_{n-m}/P_n, & n\geqslant m\geqslant 0 \\ 0, & n<m. \end{cases}$$

Then $|B|$ summability is called $|N,p_n|$ summability. If in addition $p_n=A_n^{\alpha-1}$ $(\alpha>-1)$, where

$$\sum_{n=0}^{\infty} A_n^{\alpha-1}\, x^n=(1-x)^{-\alpha} \quad (|x|<1),$$

then the $B(s_n)$ and the $|B|$ summability reduce to (C,α) $(\alpha>-1)$ mean and $|C,\alpha|$ summability, respectively.

Let σ_n^α and τ_n^α denote (C,α) mean of (s_n) and $(n\,a_n)$, respectively. Then (see [8])

(1.4) $$n(\sigma_n^\alpha-\sigma_{n-1}^\alpha)=\tau_n^\alpha.$$

A sequence (u_n) is said to be convex if and only if $\Delta^2 u_n\geqslant 0$, where Δu_n

Received Aug. 30. 1978.

$=u_n-u_{n+1}$ and $\Delta^2 u_n=\Delta(\Delta u_n)$. We shall write throughout $y_n=P_n\varepsilon_n/n^\alpha$ $(\alpha>0)$ and

$$M=\left\{(p_n)\,|\,p_n>0,\ \frac{p_{n+1}}{p_n}\leqslant\frac{p_{n+2}}{p_{n+1}}\leqslant 1\right\}.$$

2. Introduction.

The following theorems, concerning the $\left|N, \frac{1}{n+1}\right|$ summability factors, are due to Mohapatra, Das and Srivastava [10]:

THEOREM A. *Let* $0<(\varepsilon_n)\uparrow$, $0<(\lambda_n)\downarrow$ *and that*

(2.1) $$\text{(i) } \varepsilon_n\lambda_n=O(1),\quad \text{(ii) } \sum_{n=1}^{\infty} n\lambda_n|\Delta^2\varepsilon_n|<\infty.$$

Then $\sum_{n=1}^{\infty} a_n y_n \in \left|N, \frac{1}{n+1}\right|$, whenever

(2.2) $$\sum_{n=1}^{m}|\Delta\sigma_n{}^\alpha|=O(\lambda_m)\quad(\alpha>0).$$

THEOREM B. *Let* (ε_n) *be a convex sequence such that* $\sum_{n=1}^{\infty} n^{-1}\varepsilon_n$ *converges. Then* $\sum_{n=1}^{\infty} a_n y_n \in \left|N, \frac{1}{n+1}\right|$, *whenever* (2.2) *holds.*

The object of this paper is two fold: Firstly, we study, in sections 3–5, $|N, p_n|$ summability of the factored series $\sum a_n y_n$ whenever we make certain assumptions on (p_n), (ε_n) and $\sum a_n$. Secondly, in sections 6–8, we study the absolute Nörlund summability and absolute convergence factors of Fourier series.

THEOREM 1. *Let* $(p_n)\in M$ *and let* (ε_n) *satisfy the following conditions*:

(2.3) $$\begin{cases} \text{(i) } 0<(\varepsilon_n)\uparrow \text{ and bounded;} \\ \text{(ii) } \sum_{n>m} n|\Delta^2\varepsilon_n|=O(\varepsilon_m) \end{cases}$$

(2.4) $$\sum_{n=1}^{\infty}\varepsilon_n|\Delta\sigma_n{}^\alpha|<\infty\quad(\alpha>0).$$

Then $\sum_{n=1}^{\infty} a_n y_n \in |N, p_n|$ $(\alpha>0)$, *provided the following condition holds*:

(2.5) $$\sum_{n=1}^{\infty} n^{-1-\alpha}p_n<\infty \text{ for } 0<\alpha<1^{*)}.$$

* It may be noted that this condition is not required in the case when $\alpha\geq 1$.

THEOREM 2. *Let* (ε_n) *be a sequence such that* (2.3) *and* (2.4) *hold. Then* $\sum_{n=1}^{\infty} n^{-\alpha}|a_n\varepsilon_n| < \infty$ $(\alpha > 0)$.

3. Necessary lemmas

We need the following lemmas for the proof of our theorems:

LEMMA 1. *Let* $0 < (p_n)\uparrow$ *and let* (ε_n) *satisfy* (2.3). *Then*

$$y_k - y_n = O\{(n-k)k^{-1-\alpha}P_k\varepsilon_k\} \quad (\alpha > 0).$$

Proof. We have

$$y_k - y_n = \sum_{r=k}^{n-1} r^{-\alpha}P_r\Delta\varepsilon_r + \sum_{r=k}^{n-1}\varepsilon_{r+1}\{P_r\Delta(r^{-\alpha}) - (r+1)^{-\alpha}\ p_{r+1}\}$$

$$= \sum_{r=k}^{n-1} r^{-\alpha}P_r\Delta\varepsilon_r + \sum_{r=k}^{n-1} O\{r^{-1-\alpha}P_r\varepsilon_r\}, \tag{3.1}$$

and (2.3) imples that $\sum_{n=1}^{\infty}|\Delta^2\varepsilon_n| < \infty$, therefore it follows by Andersen's lemma (see Bosanquet [2]) that $\Delta(\varepsilon_n) = O(n^{-1})$. Hence

$$\sum_{r=k}^{n-1} r^{-\alpha}P_r|\Delta\varepsilon_r| \leqslant \sum_{r=k}^{n-1} r^{-\alpha}P_r\sum_{m=r}^{\infty}|\Delta^2\varepsilon_m| = \sum_{r=k}^{n-1} O\{r^{-1-\alpha}P_r\varepsilon_r\}, \tag{3.2}$$

by (2.3) (ii). Thus (3.1) and (3.2) yield the required result.

LEMMA 2. *Let* (R_n) *be defined by*

$$\sum_{n=0}^{\infty} R_n x^n = \left(\sum_{n=0}^{\infty} p_n x^n\right) \Big/ \left(\sum_{n=0}^{\infty} A_n^{\alpha-1}\ x^n\right),$$

where $|x| < 1$ *and* $0 < \alpha < 1$. *Then* $\sum_{n=0}^{\infty}|R_n| < \infty$, *whenever* (2.5) *holds.*

Proof. For $|x| < 1$, we have

$$\sum_{n=0}^{\infty} R_n x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} p_k A_{n-k}^{-\alpha-1}\right)x_n$$

so that

$$R_n = \sum_{k=0}^{n} p_k A_{n-k}^{-\alpha-1}.$$

Now splitting up the summation $\sum_{k=0}^{n}$ into $\sum_{k=0}^{m}$ and $\sum_{k=m+1}^{n}$, where m denotes the integral part of $n/2$, it may be observed that the first sum does not exceed $O\{P_n/n^{1+\alpha}\}$.

And Abel's transformation to $\sum_{k=m+1}^{n}$ yields

$$\sum_{k=m+1}^{n-1} A_{n-k}^{-\alpha} \Delta p_k + p_n A_n^{-\alpha} - p_{m+1} A_m^{-\alpha}$$

and hence we obtain that $\sum_{k=m+1}^{n}$ does not exceed $O\{P_n/n^{1+\alpha}\}$.
Thus collecting the results and using (2.5), we obtain

$$\sum_{n=0}^{\infty} |R_n| < \infty.$$

LEMMA 3 (Das[5]). *Let* $0<(p_n)\uparrow$. *Then*

$$\sum_{n=k+1}^{n} \frac{(n-k)(p_{n-k-1}-p_{n-k})}{n\ P_{n-1}} = O\left(\frac{1}{k+1}\right).$$

LEMMA 4 (Dikshit [7]). *Let* $(p_n) \in M$ *and let* $P_n = O(1)$.
Then $\sum a_n \in |N, p_n|$ *if and only if* $\sum |a_n| < \infty$.

4. Proof of Theorem 1

It is known (see Das [5], Theorem 6) that $\sum_{n=1}^{\infty} a_n y_n \in |N, p_n|$ if and only if

(4.1) $$\Sigma = \sum_{n=1}^{\infty} \frac{1}{np_n} \left| \sum_{k=1}^{n} p_{n-k}\ k a_k y_k \right| < \infty.$$

Case: $0<\alpha<1$. We have

$$\Sigma \leqslant \sum_{n=1}^{\infty} \frac{1}{np_n} \left| \sum_{k=1}^{n} p_{n-k}\ k(y_k - y_n) a_k \right| + \sum_{n=1}^{\infty} \frac{y_n}{n} |t_n^p(na_n)|$$
$$= \Sigma_1 + \Sigma_2, \text{ say,}$$

where, $t_n^p(na_n)$ denote (K, p_n) mean of (na_n). Hence

$$\Sigma_1 = \sum_{n=1}^{\infty} \frac{1}{np_n} \left| \sum_{k=1}^{n} p_{n-k}\ (y_k - y_n) \sum_{m=1}^{k} m\ A_m^{\alpha} A_{k-m}^{-\alpha-1} \Delta\sigma_{m-1}^{\alpha} \right|$$
$$\leqslant \sum_{n=1}^{\infty} \frac{1}{np_n} \sum_{m=1}^{\infty} m A_m^{\alpha}\, |\Delta\sigma_{m-1}^{\alpha}| \left| \sum_{k=m}^{n-1} \right| (\Delta_k p_{n-k})(y_k - y_n)\, A_{k-m}^{-\alpha} \Big|$$
$$+ \sum_{n=1}^{\infty} \frac{1}{np_n} \sum_{m=1}^{\infty} m A_m^{\alpha}\, |\Delta\sigma_{m-1}^{\alpha}| \left| \sum_{k=m}^{n-1} p_{n-k-1} (\Delta y_k) \right| A_{k-m}^{-\alpha}|$$
$$= \Sigma_{1,1} + \Sigma_{1,2}, \text{ say.}$$

Now, by changing the order of summation and using Lemmas 1 and 3 we obtain that

$$\Sigma_{1,1}=\sum_{m=1}^{\infty} mA_m^{\alpha}|\Delta\sigma_{m-1}^{\alpha}|\sum_{k=m}^{\infty}|A_{k-m}^{-\alpha}|\sum_{n=k+1}^{\infty}\frac{|\Delta_k p_{n-k}|}{nP_n}O((n-k)k^{-1-\alpha}P_k\varepsilon_k)$$

$$=O(1)\sum_{m=1}^{\infty} mA_m^{\alpha}|\Delta\sigma_{m-1}^{\alpha}|\sum_{k=m}^{\infty}|A_{k-m}^{-\alpha}|\frac{P_k\varepsilon_k}{k^{1+\alpha}}\sum_{n=k+1}^{\infty}\frac{(n-k)|\Delta_k p_{n-k}|}{nP_n}$$

$$=O(1)\sum_{m=1}^{\infty} mA_m^{\alpha}|\Delta\sigma_{m-1}^{\alpha}|\sum_{k=m}^{\infty}|A_{k-m}^{-\alpha}|k^{-2-\alpha}\ P_k\varepsilon_k.$$

And the inner sum does not exceed $O\{m^{-1-\alpha}\varepsilon_m\}$ therefore the boundedness of $\Sigma_{1,1}$ immediately follows from (2.4).

However, by changing the order of summation and using the estimates

$$\sum_{n=k+1}^{\infty}\frac{p_{n-k-1}}{nP_n}=O\left(\frac{1}{k}\right) \text{ and } \Delta y_k=O\left(\frac{P_k\varepsilon_k}{k^{1+\alpha}}\right)$$

the boundedness of $\Sigma_{1,2}$ may be obtained on proceeding as in $\Sigma_{1,1}$.

Finally, by the inversion formula

$$t_n^p(na_n)=\frac{1}{P_n}\sum_{k=1}^{n}R_{n-k}kA_k^{\alpha}\Delta\sigma_k^{\alpha},$$

we obtain that

$$\Sigma_2\leqslant\sum_{n=1}^{\infty}\frac{y_n}{nP_n}\sum_{k=1}^{n}|R_{n-k}|kA_k^{\alpha}|\Delta\sigma_k^{\alpha}|$$

$$=\sum_{k=1}^{\infty}kA_k^{\alpha}|\Delta\sigma_k^{\alpha}|k^{-1-\alpha}\ \varepsilon_k\sum_{n=k}^{\infty}|R_{n-k}|=O(1),$$

by Lemma 2 and (2.4). Thus, collecting the results it follows that (4.1) holds whenever $0<\alpha<1$.

Case: $\alpha\geqslant 1$. We have

$$\Sigma=\sum_{n=1}^{\infty}\frac{1}{nP_n}\left|\sum_{k=1}^{n}p_{n-k}y_k\sum_{m=1}^{k}mA_m^{\alpha}A_{k-m}^{-\alpha-1}\Delta\sigma_{m-1}^{\alpha}\right|$$

$$\leqslant\sum_{n=1}^{\infty}\frac{1}{nP_n}\sum_{m=1}^{n}m\ A_m^{\alpha}\ |\Delta\sigma_{m-1}^{\alpha}|\ \sum_{k=m}^{n}|\Delta_k p_{n-k}|\ y_k\ |A_{k-m}^{-\alpha}|$$

$$+\sum_{n=1}^{\infty}\frac{1}{nP_n}\sum_{m=1}^{n}mA_m^{\alpha}|\Delta\sigma_{m-1}^{\alpha}|\sum_{k=m}^{n}p_{n-k-1}(\Delta y_k)\,|A_{k-m}^{-\alpha}|$$

$$=R+Q, \text{ say.}$$

Now, by changing the order of summation and using the estimate

$$\sum_{n=k}^{\infty}\frac{|\Delta_k p_{n-k}|}{nP_n}=O\left(\frac{1}{kP_k}\right)$$

and condition (2.4) the boundedness of R immediately follows. The boundedness of Q may also be obtained on proceeding as in $\Sigma_{1,2}$.

This proves Theorem 1 completely.

REMARK 1. It may be observed that if, in addition to (2.3) and (2.5), (2.1) is also given then (2.2) ensures the conclusion of Theorem 1.

We now give the following corollaries of Theorem 1:

COROLLARY 1. *Let* $(p_n)\in M$ *and let* (ε_n) *satisfy* (2.3) *and* (2.5). *Then* $\sum_{n=1}^{\infty} a_n y_n \in |N, p_n|$, *whenever* $\sum_{n=1}^{\infty} a_n \in |C, \alpha|$ $(\alpha>0)$.

Proof. By hypothesis, $\sum_{n=1}^{m} |\Delta\sigma_n^{\alpha}| = O(1)$, as $m\to\infty$. Thus, by Abel's transformation

$$\sum_{n=1}^{m}\varepsilon_n\,|\Delta\sigma_n^{\alpha}| = \sum_{n=1}^{m-1}\Delta\varepsilon_n\sum_{\mu=1}^{n}|\Delta\sigma_\mu^{\alpha}| + \varepsilon_m\sum_{\mu=1}^{m}|\Delta\sigma_\mu^{\alpha}|$$

$$= O\left\{\varepsilon_m + \sum_{n=1}^{m-1}|\Delta\varepsilon_n|\right\}$$

$$= O(1), \text{ as } m\to\infty,$$

by (2.3)(i). Therefore, the proof follows from Theorem 1.

The following result generalises a result due to Mohapatra, Das and Srivastava [10]:

COROLLARY 2. *Let* $(p_n)\in M$ *and let* (2.2) *with* $\lambda_n = \log(n+1)$ *hold. Then* (2.5) *implies that*

$$\sum_{n=1}^{\infty}\frac{n^{-\alpha}a_n P_n}{\log(n+1)\,(\log\log(n+2))^{1+\delta}}\in|N, p_n| \quad (\alpha>0,\ \delta>0).$$

The case $p_n = 1/(n+1)$ of Theorem 1 leads to the following corollary which generalises a result due to Das, Srivastava and Mohapatra [6].

COROLLARY 3. *Let* (2.3) *and* (2.4) *hold. Then*

$$\sum_{n=1}^{\infty}\frac{\varepsilon_n \log(n+1) a_n}{n^{\alpha}}\in\left|N, \frac{1}{n+1}\right| \quad (\alpha>0).$$

5. Proof of Theorem 2

By Lemma 4, we know that if $(p_n)\in M$ and $(P_n)\in B$ (bounded) then the method $|N, p_n|$ is ineffective in the sense that $|N,p_n|\sim|C,0|$. Thus the extra hypothesis that $(P_n)\in B$, e. g. $p_n=\left(\frac{1}{2}\right)^n (n\geqslant 0)$, in Theorem 1 yields that

$$\sum_{n=1}^{\infty}\frac{P_n\varepsilon_n}{n^\alpha}a_n\in|C,0|$$

whenever (2.3) and (2.4) hold. Hence

$$\sum_{n=1}^{\infty}\left|\frac{a_n\varepsilon_n}{n^\alpha}\right|=\sum_{n=1}^{\infty}\frac{1}{p_n}\left|\frac{a_nP_n\varepsilon_n}{n^\alpha}\right|$$

$$\leqslant\frac{1}{p_1}\sum_{n=1}^{\infty}\left|\frac{a_nP_n\varepsilon_n}{n^\alpha}\right|<\infty.$$

6. Application to Fourier series.

We suppose that $f(t)$ is a real, periodic function with period 2π, integrable in the sense of Lebesgue. Let the Fourier series of $f(t)$, at $t=x$, be

$$\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=1}^{\infty}A_n(x),$$

where the constant term has been taken to be zero.

We write

$$\phi(t)=\frac{1}{2}\{f(x+t)+f(x-t)\}$$

$$\phi_\alpha(t)=\frac{\alpha}{t^\alpha}\int_0^t(t-u)^{\alpha-1}\,\phi(u)\,du\quad(\alpha>0),$$

$$\phi_0(t)=\phi(t),$$

$$\beta(t)=\phi_1(t)-\frac{1}{t}\int_0^t\phi_1(u)\,du.$$

We establish the following theorems:

THEOREM 3. *Let* $(p_n)\in M$ *and let* (ε_n) *satisfy* (2.3) *and* (2.5). *Then* $\phi_\alpha(t)\in BV(0,\pi)$, $\alpha<0$, *implies that*

$$\sum_{n=1}^{\infty}y_nA_n(x)\in|N,p_n|,$$

whenever (2.1) *with* $\lambda_n=\log(n+1)$ *holds.*

THEOREM 4. *Let $\beta(t)\in BV(0,\pi)$ and let (ε_n) satisfy* (2.3) *and*

$$\sum_{n=1}^{\infty} n^{-1}\varepsilon_n<\infty. \tag{6.1}$$

Then

$$\sum_{n=1}^{\infty} A_n(x)\varepsilon_n\in|C,1| \tag{6.2}$$

and

$$\sum_{n=1}^{\infty}\frac{A_n(x)\varepsilon_n\log(n+1)}{n}\in\left|N,\frac{1}{n+1}\right| \tag{6.3}$$

7. Proof of Theorem 3

We use the following lemma in the proof of the theorem:

LEMMA 5 (see [12]) *If $\phi_\alpha(t)\in BV(0,\pi)$, $\alpha\geqslant 0$. Then*

$$\sum_{n=1}^{m}|\Delta\sigma_n^\alpha(x)|=O(\log m), \tag{7.1}$$

where $\sigma_n^\alpha(x)$ is the Cesaro mean of order α of $\sum A_n(x)$.

In view of Theorem 1, it is sufficient to show (2.4) for the proof of Theorem 3. By Abel's transformation and Lemma 5, we obtain

$$\begin{aligned}\sum_{n=1}^{m}\varepsilon_n|\Delta\sigma_n^\alpha(x)| &=O(1)\sum_{n=1}^{m}\log n\,\Delta\varepsilon_n+O(\varepsilon_m\log m)\\ &=O(1)\sum_{n=1}^{m}n\log n|\Delta^2\varepsilon_n|\\ &\quad+O(m\log m\Delta\varepsilon_m)+O(\varepsilon_m\log m)\\ &=O(1)\sum_{n=1}^{m}n\log n|\Delta^2\varepsilon_n|+O(\varepsilon_m\log m)\\ &=O(1),\ \text{as } m\to\infty,\end{aligned}$$

by using $m\Delta\varepsilon_m=O(\varepsilon_m)$ and (2.1).

This completes the proof of the theorem.

REMARK 2. Theorem 3 may be compared with Bhatt [1] and in the case $p_n=1/(n+1)$ it may be compared with Mohapatra, Das and Srivastava [10].

8. Proof of theorem 4

To give a neater appearance to the proof of the theorem we work out the following order estimate in the form of a lemma:

LEMMA 6. *Let $\sigma_n^1(x)$ be $(C,1)$ mean of $\sum A_n(x)$ and let $\beta(t)\in BV(0,\pi)$. Then*

$$|\Delta\sigma_{n-1}^1(x)|=O\left(\frac{1}{n}\right).$$

Proof. We have $A_n(x)=\frac{2}{\pi}\int_0^\pi \phi(t)\cos nt\,dt.$

Integrating by parts and using the fact that $\phi_1(\pi)=0$, we obtain that

$$A_n(x)=\frac{2}{\pi}\int_0^\pi \phi_1(t)\,\{nt\ \sin nt+\cos\ nt-(\sin nt)/nt\}\,dt$$

$$-\frac{2}{\pi}\int_0^\pi \phi_1(t)\cos nt-\frac{\sin nt}{nt})\,dt$$

$$=\frac{2}{\pi}\int_0^\pi \phi_1(t)\,(nt\sin nt+\cos nt-\frac{\sin nt}{nt})\,dt$$

$$-\frac{2}{\pi}\left[\cos n\pi\int_0^\pi \phi_1(u)du-\int_0^\pi \{nt\ \sin nt+\cos nt-\frac{\sin nt}{nt}\}\frac{1}{t}\right.$$

$$\left.-\int_0^t \phi_1(u)du\right]$$

$$=\frac{2}{\pi}\int_0^\pi \beta(t)\,(nt\ \sin nt+\cos\ nt-\frac{\sin nt}{nt})dt+2\beta(\pi)\cos n\pi$$

$$=2\beta(\pi)\cos n\pi-\frac{2}{\pi}\int_0^\pi \frac{\sin nt}{nt}\beta(t)\,dt+\frac{2}{\pi}\int_0^\pi \beta(t)\,(nt\sin nt+\cos nt)dt$$

$$=-\frac{2}{\pi}\left[\int_0^{\pi/n}\frac{\sin nt}{nt}\beta(t)dt+\int_{\pi/n}^\pi \frac{\sin nt}{nt}\beta(t)dt\right]-\frac{4}{n\pi}\int_0^\pi \sin nt\,d\beta(t)$$

$$+\frac{2}{\pi}\int_0^\pi t\ \cos nt\,d\beta(t)$$

$$=O\left(\frac{1}{n}\right)+\frac{2}{\pi}\int_0^\pi t\ \cos nt\,d\beta(t),$$

by the hypothesis. Hence

$$|\Delta\sigma_{n-1}^1(x)|=\frac{1}{n^2}\left|\sum_{k=1}^{n} k\ A_k(x)\right|=O\left(\frac{1}{n}\right)+\frac{2}{\pi n^2}\left|\int_0^\pi t\ d\beta(t)\sum_{k=1}^{n} k\ \cos kt\right|$$

$$=O\left(\frac{1}{n}\right)+\frac{2}{\pi n^2}\int_0^\pi t\,|d\beta(t)|\,O\left(\frac{n}{t}\right)=O\left(\frac{1}{n}\right),$$

since $\beta(t)\in BV(0,\pi)$. Consequently, this completes the proof of the lemma.

Now application of Lemma 6 to the cases $p_n=\frac{1}{n+1}$ and $p_n=1$, for all n, of Theorem 1 for $\alpha=1$ yield results (6.2) and (6.3) respectively.

This completes the proof of Theorem 4.

REMARK 3. It may be observed that $\phi_1(t)\in BV(0,\pi)$ implies that $\beta(t)\in BV(0,\pi)$ but converse is not true. For example, if $\phi_1(t)=\log 2\pi/t$ then $\phi_1(t)\in BV(0,\pi)$ while $\beta(t)=1$, which is of $BV(0,\pi)$. Thus (6.2) and (6.3) respectively, improve results due to Cheng [4] and Lal [9]. However, Cheng's result was also improved by Chandra [3] in another direction and it is interesting to investigate the relationship between the two conditions:

$$\beta(t)\in BV(0,\pi) \text{ and } \phi_1(t)(\log k/t)^{-\delta}\in BV(0,\pi),\ \delta>0,$$

the later condition was taken by Chandra [3] to improve Cheng's result.

References

1. Bhatt, S. N., *On the absolute summability of factored Fourier series*, Annali di Matematica Pura ed Applicata, Serie IV Tomo LXXII (1966), 253–266.
2. Bosanquet, L. S., *Note on Bohr–Hardy theorem*, Jour. London Math. Soc. **17** (1942), 166–173.
3. Chandra, P., *Absolute summability factors for Fourier series*, Rendiconti Accademia Nazionale dei XL (4), 24–25 (1974), 3–23.
4. Cheng, M. T., *Summability factors of Fourier series at a point*, Duke Math. Jour. **15** (1948), 29–36.
5. Das, G., *Tauberian theorems for absolute Nörlund summability*, Proc. London Math. Soc. (3), **19** (1969), 357–384.
6. Das, G., Srivastava, V. P. and Mohapatra, R. N., *On absolute summability factors of infinite series*, Jour. Indian Math. Soc. **31** (1967), 189–200.
7. Dikshit, G. D., *A note on Riesz and Nörlund means*, Rendicontidel Circolo Matematico di Palermo (2), **18** (1969), 49–61.
8. Kogbetliantz, E., *Sur les series absolument sommables par la méthode des moyennes arithmétique*, Bull. Sci. Math. (2) **49** (1925), 234–256.
9. Lal, S. N., *On the absolute Harmonic summabilty of the factored Fourier series*, Proc. American Math. Soc. **14** (1963), 311–319.
10. Mohapatra, R. N., Das, G. and Srivastava, V. P., *On absolute summability factors of infinite series and their application to Fourier series*, Proc. Camb. Phil. Soc. **63** (1967), 107–118.
11. Russell, D. C., *Note on inclusion theorems for infinite matrices*, Jour. London Math. Soc. **33** (1958), 50–62.
12. Sunouchi, G., *On the absolute summability factors*, Kōdai Math. Seminar Report, **1** (1954), 59–62.

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