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TOPOLOGY OF LONG LINE AND INSEPARABLE CONNECTED MANIFOLDS[1)]

BY JEHPILL KIM

1. Introduction

This note starts by proving a condition under which manifolds embedded in L^n, the n-fold product of the long line, fail to have boundary collars. After recognizing that pretty many inseparable manifolds have boundary not collared, we construct uncountably many line bundles over L^n, implying in particular that there are uncountably many simply connected open 2-manifolds.

In view of Brown [1], failure of boundary collaring is an essential difference between nonparacompact manifolds and metric ones. A typical counter example for the boundary collar problem is the Prüfer manifold [5, 8], a separable Hausdorff manifold with uncountably many boundary components. We do not know, however, if any Hausdorff manifold is known to have connected boundary that is not collared. Existence of uncountably many simply connected open manifolds is also in sharp contrast compared with metric manifolds since the plane $\mathbf{R}^2$ is the only simply connected open 2-manifold that is metrizable. Note incidentally that there are uncountably many contractible manifolds of dimension higher than 2 by [2], [7] and [3].

Although we do not require paracompactness, *we shall treat only those manifolds which are normal Hausdorff spaces*. If M is a manifold, ∂M and $\mathring{M}$ will denote the boundary and interior of M. A subset of a manifold M is said to be *bounded* if it is contained in a compact subset of M.

The Stone-Čech compactification of a Tychonov space X is dneoted by βX. If f is a map between Tychonov spaces, f_* will denote the Stone extension of f. We do not discriminate points and singletons notationally; x can stand for $\{x\}$ not just only the point x.

2. Definition and basic properties of the long line

The *long ray* is the set $L_+ = W\times[0, 1]$ ordered lexicographically, where W denotes all countable ordinals. Let L_- be an order reversing copy of L_+

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and identify the first element of L_+ with last element of L_-. The resulting ordered space L is called the *long line*. L is a connected 1-manifold because every bounded closed interval of L is separable and irreducibly connected between end points. Note that countable sets are bounded in L and that a subset of L is metrizable exactly when it is bounded.

For the sake of convenience, designate the unique common point of L_+ and L_- by 0, and let $-x$ denote the image of x under the obvious order reversing involution of L fixing 0. Following the customary usage for real numbers, we denote by $|x|$ one of the two points $x, -x$ that belongs to L_+.

Any homeomorph of L or L_+ will also be called a long line or a long ray. A homeomorph of L_+ contained in a long line or a long ray is often called a *tail*. The following well known result play the key role.

LEMMA 1. *If A and B are cofinal and closed in L_+, then so is $A \cap B$.*

Immediate corollaries are:

(i) *$\beta L_+ - L_+$ consists of one point, which we shall denote by Ω. Accordingly, we may regard $\beta L = L \cup \{\Omega, -\Omega\}$ to be an ordered space by asking $-\Omega < x < \Omega$ for all x in L.*

(ii) *Every map of L_+ into a metric space is constant on some tail.*

Here and afterwards, maps are assumed to be continuous.

LEMMA 2. $\beta(L_+ \times I) = \beta L_+ \times I$. *In particular, $\beta(L_+ \times I) - (L_+ \times I)$ is an arc.*

LEMMA 3. $\beta L^n = (\beta L)^n$.

These two lemmas follow from Glicksberg [4]. As usual, I is the closed unit interval $[0, 1]$.

By Lemma 3, we may write points of βL^n as $x = (x^1, \cdots, x^n)$, $x^i = p_i(x)$, where $p_i \colon \beta L_n = (\beta L)^n \to \beta L$ are projections. By a *vertex* of L^n, we shall mean any of those 2^n points in βL^n having all coordinates in $\beta L - L$. The *diagonal map* for L^n is the unique map $D \colon L \to L^n$ such that $p_i \circ D$ is the identity map of L for all i.

LEMMA 4. *If A is a closed subset of $L_+{}^n$ such that $p_i(A)$ is unbounded for each i, then A meets every tail of $D(L_+)$.*

LEMMA 5. *L^n is normal.*

For Lemma 4, observe that the sets $p_i(A)$ meet in a cofinal subset of L_+ by Lemma 1. For Lemma 5, let A, B be closed subsets of L^n sharing a limit point x in $\beta L^n - L^n$ and conclude that A meets B in L^n. If $n=1$, this is

Lemma 1. If x is a vertex, $n>1$, use Lemma 4. If x is not a vertex, pick an i with $x^i=p_i(x)$ in L and observe that both $A\cap p_i^{-1}(x^i)$ and $B\cap p_i^{-1}(x^i)$ meet every neighborhood of x. Then use induction to complete the proof.

At this stage, we can answer a problem of Hirsch [5. p. 118] in the following form. A different proof is obtained in [6] as a byproduct of homotopy classification and fixed point theory for self maps of L_+, without using Stone-Čech compactifications.

THEOREM 1. *There does not exist an embedding* $h: L\times I\to L^n$ *such that* $h(x,0)=D(x)$ *for all* x *in* L_+.

Proof. If there is an h as in the theorem, we have $h_*(\Omega,1)\neq h_*(\Omega,0)$ because h_* must be an embedding by Lemma 5. Hence $p_i\circ h_*(\Omega,1)\neq p_i\circ h_*(\Omega,0)=p_i\circ D_*(\Omega)=\Omega$ for some i. It follows that $\alpha=p_i\circ h_*(\beta(L_+\times I)-(L_+\times I))$ is a nondegenerate subset of βL containing Ω. This is impossible since α must be a continuous image of an arc by Lemma 2.

3. A cellular structure on L^n

An *inseparable cell complex of dimension n*, or briefly ***n*-complex**, is a finite ascending sequence of spaces M_k, $0\leq k\leq n$, with M_0 totally disconnected and M_n not separable such that each component of M_k-M_{k-1}, $k>0$, is locally k-eu clidean, conditionally compact in M_k, and homotopy connected in all dimensions in the sense that all homo-topy groups of if vanish. Components of M_k-M_{k-1} are called *k-pseudocells*, or simply *k-cells* if no confusion is likely: M_k is called the *k-skeleton* of the complex. If M is an n-manifold contained in the top dimensional skeleton M_n and if M_n-M consists of all those points of $\beta M-M$ that are limit point of some long ray in M, then we may associate this n-complex structure to M. In this case, cells and skeletons of M are those of the associated complex although they need not be subspaces of M in general. Very frequently, we shall express this situation by saying that M is a *cellular manifold* assuming the associated cell complex structure tacitly.

The space L^n has a structure very similar to the obvious cell complex structure of the unit n-cube I^n. A k-cell σ_e consists of those points x in βL^n with exactly $n-k$ coordinates $p_i(x)=e(i)$ in $\beta L-L$, where e is a function into $\beta L-L$ defined on a set of $n-k$ integers between 0 and $n+1$. L^n has exactly $2^{n-k}\binom{n}{k}$ cells of dimension k, all of which are homeomorphs of L^k. Of course, the k-skeleton X^k of L^n is the union of cells of dimension not exceeding k; X_{k-1} consists of those points of X_k at which X_k is not locally separable.

LEMMA 6. *Pseudocells of L^n are path components of βL^n.*

Proof. If σ_e and $\sigma_{e'}$ are distinct cells, then there is an i such that $e(i)\neq e'(i)$ or i belongs to the domain of exactly one of e, e'. Because βL has exactly three path components, L, Ω and $-\Omega$, this means that the projection p_i maps σ_e, $\sigma_{e'}$ into distinct path components of βL.

REMARK. We have shown in [6] that self maps of L_+ are divided into two homotopy classes, bounded maps and unbounded ones. Lemma 7 reflects this; every path in $\beta L^n - L^n$ is obtainable as end point trace of suitable one parameter family of long rays in L^n. In this regard, maps of L_+ into L^n are divided into $\sum_{k=0}^{n} 2^{n-k}\binom{n}{k} = 3^n$ homotopy classes, each determined by cells of L^n. Any of these homotopy classes except the classs consisting of bounded maps can be represented by a smooth long ray. Accordingly, there are $3^n - 1$ homotopically inequivalent long rays in L^n such that every long ray in L^n is homotopic with one of them. Of course, some of them are equivalently embedded; there are only n inequivalent embeddings among them because faces of L^n are classified by dimensions under self homeomorphisms of L^n. One should also be aware that there are many other inequivalent embeddings because wild embeddings exist for $n>2$.

A map between cellular manifolds is called a *cellular map* if it sends pseudocells into pseudocells. Terms like *cellular isomorphism* and *cellular submanifold* will have obvious meaning as categorical terms. Note that every subdivision of associated cell complex of a cellular manifold gives rise to a cellular submanifold in this definition. A noteworthy consequence of Lemma 6 is: "If a cellular manifolds is topologically embedded in L^n, then it is a cellular submanifold."

THEOREM 2. *Let M be a cellular submanifold of L^n. If a k-pseudocell σ of M meets the k-skeleton X_k of L^n, $k<n$, then there does not exist an embedding $h: M\times I \to L^n$ such that $h(x, 0) = x$ for all x in M.*

Proof. Let τ be the cell of L^n containing σ. Pick a point y in σ and let T be a long ray in M of which y is a limit point. If there is an h as in the theorem, it follows from Lemmas 2, 5 and 6 that $\alpha = h_*(y\times I) = h_*(\beta(T\times I) - (T\times I))$ is an arc in τ and that the open arc $\alpha - y$ fails to touch σ. This is impossible since τ must have dimension k by Lemma 6, and so σ contains a $(k-1)$-sphere sparating y form $\tau - \sigma$.

Let M be a manifold and let y be a limit point of ∂M contained in $\beta M - M$. We shall say that ∂M is *locally collard at y* if there is an open neighborhood U in βM of y such that there is an embedding $h: (\partial M\cap U)\times I \to M\cap U$ with $h(x) = x$ for all x in $\partial M \cap U$. If U can be taken to be whole of

βM, ∂M is collared in usual sense.

THEOREM 3. *For each* $n\geq 1$, *there exists a manifold* M *of dimension* $n+1$ *such that* (i) ∂M *is not locally collared at any limit point of* ∂M *in* $\beta M-M$, *and* (ii) M *and* ∂M *are homotopy connected in all dimensions.*

Proof. Let M consists of those points x of L^{n+1} with $x^{n+1}\geq|x^i|$ for all $i\leq n$. The first asseration follows from Lemma 5 and Theorem 2 above. For the second assertion, observe that each a-sublevel set $x^{n+1}\leq a, a\in L_+$, meets M and ∂M in contractible sets. In fact, the intersections are homeomorphs of I^{n+1}, I^n if $a>0$. Since compact sets must be bounded in M, this shows that both $\pi_k(M)$ and $\pi_k(\partial M)$ are trivial for all $k\geq 0$.

REMARK. If M' is a metric manifold of dimension $n+1$, $n\geq 1$, and if x is a point of $\partial M'$, there is a Hausdorff manifold M'' with $\partial M''$ not collared such that $\pi_k(M'')\approx\pi_k(M')$ and $\pi_k(\partial M'')\approx\pi_k(\partial M'-x)$ for all k. Such M'' can be most easily obtained by sewing a small n-cell of $\partial M'$ containing x homeomorphically, to an n-cell in ∂M with M as in Theorem 3. In fact, we have uncountably many such M'' by the technique that will be developed in the next section.

4. Uncountably many line bundles over L^n

We begin this section by describing two nontrivial line segment bundles H and K that will be used to define a class of cellular manifolds all are line bundles over L^n. Let $a>0$ be an element of L and let H consists of those x in L^{n+1} such that $|x^{n+1}|\leq a$ or $|x^{n+1}|\leq|x^i|$ for all $i\leq n$. Then define K to be the part of H lying in the a-sublevel set $x^{n+1}\leq a$. H has two boundary components, the upper boundary $\partial_+H=\partial H\cap p_{n+1}{}^{-1}(L_+)$ and the lower boundary $\partial_-H=\partial H\cap p_{n+1}{}^{-1}(L_-)$. The upper and lower boundaries of K are $\partial_+K=L^{n+1}\cap p_{n+1}{}^{-1}(a)$ and $\partial_-K=\partial_-H$. Since Lemma 5 implies that H, K are normal and that their closures in βL^{n+1} agree with Stone-Čech compactifications, we may regard H and K as cellular manifolds by specifying their cells; pseudocells of H are $\mathring{H}$ and path components of $\beta H-\mathring{H}$; those of K are path components of $\beta L^n\times a$ and those part of cells of H lying below the a-level $x^{n+1}=a$.

LEMMA 7. H *and* K *are bundless over* L^n *with each fibre homeomorphic with* I. *Moreover, they are nontrivial and inequivalent each other.*

Proof. Let $q\colon L^{n+1}\to L^n$ be defined by $p_i\circ q(x)=p_i(x)$ for $i\leq n$. Since $H\cap q^{-1}(X)$ and $K\cap q^{-1}(X)$ are homeomorphic with $X\times I$ for all bounded subsets X of L^n, H and K are I-bundles whose projections are q suitably restricted. They are nontrivial and inequivalent each other because no boundary component of H is collared while K has exactly one boundary component collared in K, by virtue of Theorem 2.

Now to the proposed construction of uncountably many bundles: If $c=(c_1, \cdots, c_j, \cdots)$ is an infinite dyadic sequence, i.e., if $c_j=1$ or 0, let M_j be a copy of H or K according to whether c_j is 1 or 0. For $j\leq 0$, we let $M_j=L^n\times[j-1, j]$. The boundary components ∂_+M_j, ∂_-M_j and bundle projections $q_j: M_j\to L^n$ are defined in the obvious fashion for all integers j. Form the disjoint union of all M_j and sew ∂_+M_j to ∂_-M_{j+1} by identifying each pair of points having the same image under bundle projections. The resulting space M is a line bunddle over L^n if we define the projection $q: M\to L^n$ by requiring that q and q_j agree on each M_j. Moreover, we may regard M as a cellular manifold because every building block M_j is associated with an $(n+1)$-complex structure in the obvious manner. To verify this is indeed the case, we need to show that every long ray in M has its end point in some βM_j and that M is normal. For the first assertion, observe that a long ray is countably compact and it can touch only finitely many βM_j. For the second, let A be a closed subset of M and let f be a real valued map defined on A. Since each M_j is normal by Lemma 5, we can successively extend f to maps defined on $A\cup M_{-j}\cup\cdots\cup M_j$ to get a continuous extension of f defined on M. Thus M has a well defined cell complex structure. Denote by X_k the part of the k-skeleton of M contained in $\beta M-M$. Also let Y_k be the part of X_k whose points are limit point of M_j for some $j>0$. It is clear that the X_k's and Y_k's give rise to inseparable n-complex structures for the spaces $X=X_n$ and $Y=Y_n$.

Now suppose c, c' are infinite dyadic sequences and let M, M' be constructed as above corresponding to c, c', respectively. The cell complexes X, X' and Y, Y' are also defined as above.

LEMMA 8. *If $h: M\to M'$ is a homeomorphism, its Stone extension h_* is an isomorphism of inseparable complexes between Y and Y'.*

Proof By definition, the complement of a cellular manifold in its top skeleton does not depend on the associated cell complex because its points are precisely those which can be accessible by long rays in the manifold. Hence the Stone extension h_* maps X homeomorphically onto X'. In particular, h_* sends path components of X to path components of X'.

Next observe that each path component of X, X' contained in Y, Y' is homeomorphic with L^k or $L^k\times L_-$ for suitable $k\geq 0$. Path components of X, X' that are not contained in Y, Y' are of the form $\sigma\times(-\infty, 0]$, σ being any cell of $\beta\partial_+M_0-\partial_+M_0$ or $\beta\partial_+M'_0-\partial_+M'_0$. Since h_* must be a homeomorphism on each $\sigma\times(-\infty, 0]$ which is a manifold whose boundary is $\sigma\times 0$, it follows that h_* sends $X-Y$ onto $X'-Y'$ and $\beta\partial_+M_0-\partial_+M_0$ onto $\beta\partial_+M'_0-\partial_+M'_0$. In turn, h_* sends Y onto Y'.

Now, for each $k<n$, the k-skeleton Y_k of $Y=Y_n$ consists of those points of Y_{k+1} at which Y_{k+1} is not locally separable. Since the same is true for all skeletons Y'_k, $k<n$, of $Y'=Y'_n$, h_* sends Y_k to Y'_k for all $k\leq n$. This implies that both h_* and h_*^{-1} are cellular maps between Y, Y'.

THEOREM 4. *If there is a homeomorphism* $h:M\to M'$, *then* $c=c'$.

Proof. For each vertex v of Y, there is a unique vertex v_0 of ∂_+M_0 and an integer $j\geq 0$ such that v is the point of intersection of the fibre $q_*^{-1}(v_0)$ with the section $\beta\partial_+M_j-\partial_+M_j$, where q_* is the Stone extension of the bundle projection $q:M\to L^n=\partial_+M_0$. Vertices of Y' are also uniquely expressed in a similar fashion. We shall use this to prove that h_* sends the vertex of ∂_+M_j contained in $q_*^{-1}(v_0)$ to the vertex of $\partial_+M'_j$ contained in $q'_*{}^{-1}(h_*(v_0))$ for all integers $j\geq 0$ and for all vertices v_0 of ∂_+M_0. Since this is true for $j=0$ by Lemma 8, assume true for j and we shall prove the case $j+1$. Let v_{j+1} be any vertex of ∂_+M_j with $q_*(v_{j+1})=v_0$ and let v_j be the unique vertex of ∂_+M_j in $q_*^{-1}(v_0)$. The edge(=1-pseudosimplex) σ of Y having v_j, v_{j+1} as vertices has the property that σ is the only edge of Y having v_j as a vertex such that the other vertex of σ is not in any of $\beta\partial_+M_0-\partial_+M_0, \cdots, \beta\partial_+M_j-\partial_+M_j$. Since h_* is a cellular isomorphism on Y, the induction hypothesis implies that $h_*(\sigma)$ is the only edge of Y' with $h_*(v_j)$ as a vertex such that the other vertex of $h_*(\sigma)$ is not in any of $\beta\partial_+M'_0-\partial_+M'_0, \cdots, \beta\partial_+M'_j-\partial_+M'_j$. This shows that $h^*(v_{j+1})$ lies in $q'_*{}^{-1}(h_*(v_0))$ and $\beta\partial_+M'_{j+1}-\partial_+M'_{j+1}$, and the induction is complete.

The above argument shows that h_* is a cellular isomorphism of $Y\cap q_*^{-1}(v_0)$ to $Y'\cap q'_*{}^{-1}(h_*(v_0)$ for vertices v_0 of ∂_+M_0. In turn, h_* becomes a homeomorphism of $q^{*-1}(v_0)\cap\beta M_j$ to $q'_*{}^{-1}(h_*(v_0))\cap\beta M'_j$ for all $j\geq 1$. But $q_*^{-1}(v_0)\cap\beta M_j$ is homeomorphic with βL or βL_+ according as c_j is 1 or 0 and similarly for $q'_*{}^{-1}(h_*(v_0))\cap\beta M'_j$, and it follows that $c_j=c'_j$ for all $j\geq 1$ if there is a homeomorphism $h:M\to M'$.

In light of Theorem 4, we have

COROLLARY 1. *There are uncountably many inequivalent line bundles over* L^n *such that their bundle spaces are pairwise non-homeomorphic.*

COROLLARY 2. *There are uncontably many topologically distinct noncontractible open* n-*manifolds*, $n\geq 2$, *which are homotopy connected in all dimensions.*

These results are sources of many attractive counter examples: Let, for example, M_+ be the part of M in Theorem 4 that corresponds to all M_j with $j\geq 0$ and let M^* be the part corresponding to all M_j, $j\geq 1$. The M^* are uncountably many examples of manifolds without boundary collars. If we double M_+, we have uncountably many line bundles with obvious reflecitons about the 0-section $\partial_- M_0 = L^n \times 0$. Since the part corresponding to $(\beta\partial_- M^0 - \partial_- M^0)\times[-1, 1]$ is recognizable by its path components, we can discriminate topological types by the techique used in proving Lemma 8 and Theorem 4. In fact, we can go one step further. Take M_j for all countable ordinals j and sew them along boundaries. Proceeding as above, we have, we have $2^{\aleph}$ distinct long line bundles over L^n as well as $2^{\aleph}$ distinct open manifolds that cannot be discriminated by homotopy groups, where $\aleph$ denotes the first uncountable cardinal. Proof is the same as the case of line bundles except that we use transfinte induction instead of mathematical induction. We omit the details.

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Seoul National University