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RESTRICTION OF POINCARÉ'S OPERATOR AND BOUNDEDNESS OF INTEGRABLE FORMS.

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ABSTRACT. We shall consider an operator which is a restriction of the Poincaré's operator but still the operator gives all of the integrable forms in some cases. As an application of this operator, we investigate the conjecture that integrable forms are bounded and the inclusion of integrable forms into the bounded forms is continuous.

Let $\Delta=\{z:|z|<1\}$ be the unit disk of the complex plane. Then the set of conformal automorphisms of Δ forms a group. Let G be a discontinuous subgroup of the automorphism group of Δ, then it is called as a Fucshian group. Let $\lambda(z)=(1-|z|^2)$, then $(\lambda(z)^{-1}|dz|)$ is called as the Poincaré metric on Δ. Let w be a simply connected set in Δ that satisfies $\Delta=\bigcup\{A(w):A\in G\}$, $A(w)$ and $B(w)$ has no common part for different elements A and B in G. It is well known that every Fucshian group G has such a set w and we call w a fundamental domain of the group G.

For the details of the fundamental domain and Fucshian group consult [3].

Let G be a Fucshian group on Δ, for A in G let $A'(z)$ be the derivation of the complex function $A(z)$ and $q\geq 2$ be a fixed integer.

A *measurable G-automorphic form* of weight $(-2q)$ is a measurable function on Δ which satisfies

$$\mu(A(z))A'(z)^q=\mu(z) \text{ for all } A \text{ in } G.$$

A measurable G-automorphic form μ is called *integrable* if

$$(1) \qquad |\mu|=\int_w \lambda(z)^{2-q}|\mu(z)|dxdy<\infty,$$

and it is called a *bounded* form if

$$(2) \qquad |\mu|=\sup_{z\in w}|\lambda(z)^{-q}\ \mu(z)|<\infty.$$

Let $L^\infty(q,G)$ be the set of all measurable G-automorphic forms on Δ which satisfies (2). Then $L^\infty(q,G)$ is a Banach space with the norm given by (2). Let $L(q,G)$ be the set of all integrable automorphic forms on Δ which

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satisfy the condition (1). Then it is also a Banach space with the given norm.

All analytic functions in $L(q, G)$ and $L^\infty(q, G)$ form closed subspaces which we shall denote by $A(q, G)$ and $B(q, G)$, respectively. For a measurable function μ on Δ, let

$$(\beta\mu)(z) = (2q-1)\pi^{q-1}\int_\Delta \lambda(\zeta)^{2-2q}K(z,\zeta)^q\mu(\zeta)\,dxdy$$

where $k(z,\zeta) = [\pi(1-z\bar{\zeta})^2]^{-1}$.

Then it is known β to be a bounded projection of $L(q, G)$ onto $A(q, G)$ and $L^\infty(q, G)$ onto $B(q, G)$. We shall denote the trivial group by I. For an analytic function ϕ in $A(q, I)$, one defines the Poincaré's operator θ as the following:

$$(\theta\phi)(z) = \sum_{A\in G}\phi(A(z))A'(z)^q.$$

The operator θ maps $A(q, I)$ onto $A(q, G)$ and it is a bounded linear map. Let χ_w be the characteristic function of the fundamental domain w and let R be the space of all analytic functions f on w which satisfy

$$\int_w |f(z)|\lambda(z)^{2-q}dxdy < \infty.$$

Then R forms a Banach space with the above norm.

LEMMA 1. *For any $\phi\in R$, let E be the extension of ϕ on Δ by the requirement $\phi(A(z))A'(z)^q = \phi(z)$. Then $\beta\circ E$ maps R boundedly onto $A(q, G)$ and it satisfies $(\beta\circ E)(\chi_w\cdot g) = g$ for g in $A(q, G)$.*

Proof. By the definition of E it is clear that $E(\phi)$ is a G-integrable form. Hence $\beta\circ E$ maps R into $A(q, G)$. On the other hand, if g is an element of $A(q, G)$, then $\chi_w g$ is contained in R. Since $E(\chi_w g)$ is g and β is the identity operator on the analytic forms, we conclude the desired results.

Now define the bounded map χ on $L(q, I)$ by

$$(\chi f)(z) = (\chi_w\cdot f)(z) \text{ for } f \text{ in } L(q, I).$$

Since R is a subspace of $L(q, I)$, one observe that the composition map $\beta\circ E\circ\chi$ maps $L(q, I)$ to $A(q, G)$ and it is a surjection. We shall use the notation α for the map $\beta\circ\chi\circ E$.

LEMMA 2. *The map α is a bounded operator which maps $L(q, I)$ onto $A(q, G)$.*

We note that for any element ϕ in $A(q, G)$, both α and the operator θ maps $\chi_w\phi$ to ϕ. In [3], Kra considered α on $B(q, I)$ and observed that it maps the space onto $B(q,G)$.

In the following we investigate the operator α and as an application we examine the conjecture that the integrable forms are bounded. If the dimension of integrable forms $A(q,G)$ is finite the conjecture is well known and for the infinite case Lehner gives partial solution, see [2], [3] or [1].

PROPOSITION 1. *The map α is a bounded operator which maps $A(q,I)$ into $A(q,G)$ and has the following properties;*

(i) *the range of α is dense in $A(q,G)$,*

(ii) *if the range of α is closed, then the inclusion of $A(q,G)$ into $B(q,G)$ is continuous, and*

(iii) *if $A(q,G)$ has a finite dimension, then α is a surjection of $A(q,I)$ onto $A(q,G)$.*

Proof. It is known that the polynomials are dense in the space of integrable forms for trivial group. For the proof of this theorem we need the polynomial density in the case $q<1$ on simply connected domain (see [2] or [1]). Applying the above fact to w, we conclude that the polynomials are also dense in R. Since $A(q,I)$ contains all polynomials and $\beta\circ E$ maps R onto $A(q,G)$, it is clear that the image of α is dense. For the proof (ii), consider the map β on $L(q,I)$. Let ϕ be an element of $L(q,I)$. Then the norm of $\beta(\phi)$ in $L^{\infty}(q,I)$ is given by $\sup|\lambda(z)^{-q}\beta(\phi)(z)|$. We can estimate the norm by the following equation:

$$|\lambda(z)^{-q}\beta(\phi)(z)|=(2q-1)\pi^{q-1}\int_{\Delta}\lambda(z)^{-q}\lambda(\zeta)^{2-2q}K(z,\zeta)^{q}\mu(\zeta)\,dxdy$$

$$=|(2q-1)\pi^{q-1}\int_{\Delta}\lambda(z)^{-q}\lambda(\zeta)^{-q}K(z,\zeta)^{q}[\lambda(\zeta)^{2-2q}\mu(\zeta)]\,dxdy.$$

Since $|\lambda(z)^{-1}\lambda(\zeta)^{-1}K(z,\zeta)|\leq 4$, we have $|\beta(\phi)|\leq 4\cdot c\cdot\|\phi\|$ for some constant c. Hence β is a bounded map on $L(q,I)$ which maps it into $B(q,I)$. Since β is the identity on $A(q,I)$, we conclude that $A(q,I)\subseteq B(q,I)$. We observed that α is also defined on $B(q,I)$ and maps it boundedly onto $B(q,G)$. We have the following diagram;

$$\begin{array}{ccc} B(q,I) & \xrightarrow[\alpha]{} & B(q,G) \\ \uparrow i & & \\ A(q,I) & \xrightarrow[\alpha]{} & A(q,G) \end{array}$$

Since α maps $A(q,I)$ onto a dense subset of $A(q,G)$, if we assume that the image is closed then it must be a surjection. Let N be the kernel of α on the integrable forms of the trivial group $(A(q,I))$. Then we have an isomorphism

$A(q,I)/N$ onto $A(q,G)$. Since $\alpha\circ i=\alpha$, we conclude that the inclusion $A(q,G)$ into $B(q,G)$ must be continuous. Since the image of α is dense in the space of analytic integrable forms of G, by an application of (ii) we have (iii).

In the above proof we note that if the dimension of $A(q,G)$ is finite then the inclusion of integrable forms into the bounded forms is continuous, which is well known see ([4] or [3]).

COROLLARY 1. *If the dimension of analytic integrable forms are finite then the inclusion* $A(q,G)$ *into* $B(q,G)$ *is continuous and it can be given by* $\alpha\cdot i\cdot\alpha^{-1}$.

Now we consider the conjugate map of α and we give some information about the map $\beta\circ\chi$.

THEOREM 2. *The following statements are equivalent;*
(i) α *maps* $A(q,I)$ *onto a closed subspace of* $A(q,G)$,
(ii) $\beta\circ\chi$ *maps* $B(q,G)$ *onto a closed subspace of* $A(q,I)$, *and*
(iii) $\beta\circ\chi$ *is a homeomorphism.*

Proof. Let α^* be the conjugate map of α. Then for any element ν of $L^\infty(q,G)$ and μ of $L(q,G)$ we define the petterson scalar products

$$(\mu,\nu)_G=\int_w \lambda(\zeta)^{2-2q}\mu\bar{\nu}dxdy.$$

It is well known that the above product gives dualities of $L(q,G)$ and $A(q,G)$ to $L^\infty(q,G)$ and $B(q,G)$, respectively (see [3] or [1]). By a simple calculation we have

$$(\beta\mu,\nu)=(\mu,\beta\nu). \tag{3}$$

For the details of the above, see [3]. Using this and the fact that β is the identity on analytic forms, we conclude that $\alpha^*(\phi)$ can be identified with the analytic function $(\beta\circ\chi)(\phi)$ for any $\phi\in B(q,G)$. Hence we have the equivalence of (i) and (ii).

To establish the equivalence of (iii) and (ii), it suffices to prove that $(\beta\circ\chi)$ is an injection. We first consider α as a map of $L(q,I)$ to $A(q,G)$. Then for any element $\phi\in B(q,G)$ we see that $\alpha^*(\phi)$ as an element of $L^\infty(q,I)$ has the same norm with ϕ. Also note that α^* is an injection as a map of $B(q,G)$ into $L^\infty(q,I)$. Now restrict the map on $A(q,I)$, then the norm of $\alpha^*(\phi)$ may decrease but by the identity (3), we easily conclude that still α^* is injective.

In the above proof we noted that $\beta\circ\chi$ is a one-to-one map.

COROLLARY 2. *The map* $\beta\circ\chi$ is a one-to-one map of $B(q,G)$ into $A(q,I)$.

If the dimension of $A(q, G)$ is finite, then the map is a homeomorphism.

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