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CHARACTERIZATIONS OF SELF-DECOMPOSABLE PROBABILITY MEASURES ON LCTVS.

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1. Introduction.

The purpose of this paper is threefold: Firstly, we obtain several characterizations of K-regular self-decomposable probability (prob.) measures on real locally convex Hausdorff topological vector space (LCTVS) E. These results are motivated from and extension of, similar known characterizations due to Levy [9] and Kumar and Schreiber [8], of self-decomposable prob. measures defined, respectively, on Euclidean spaces and real separable Banach spaces. Secondly, we prove that the K-regular weak limit of K-regular self-decomposable prob. measures on E is self-decomposable. Thirdly, we show that the topological support of a K-regular, symmetric self-decomposable prob. measure on E is a (closed) subspace of E.

2. Notation and preliminaries.

The letters E and E' will denote a LCTVS and its topological dual, respectively. R and R^+ will denote the reals and positive reals, respectively. By a prob. measure on E we will always mean that it is defined on $\mathfrak{B}(E)$, the smallest σ-algebra containing the open sets of E. A prob. measure μ is *K-regular* if $\mu(B)=\sup_{K\subset B}\mu(K)$ for every $B\in\mathfrak{B}(E)$, where K ranges over the compact subsets of B. $M_K(E)$ will denote the set of all K-regular prob. measures on E. If μ, $\nu\in M_K(E)$, the *convolution* of μ and ν is defined by

$$\mu*\nu=\int_E \mu(B-x)\nu(dx),$$

for every $B\in\mathfrak{B}(E)$ [2]. With the topology of weak convergence of measures, and convolution as a multiplication, $M_K(E)$ becomes an abelian topological semigroup [2].

Let $\langle\ ,\ \rangle$ be the natural bilinear form on $E\times E'$. For a prob. measure μ on E, the *characteristic functional* of μ, denoted by $\hat{\mu}(\cdot)$, is the function on E' defined by

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$$\hat{\mu}(x')=\int_E e^{i<x,x'>}\mu(dx)$$

for every $x'\in E'$. It is easily seen that, for every $\mu,\nu\in M_K(E)$, $\widehat{\mu*\nu}(\cdot)=\hat{\mu}(\cdot)\hat{\nu}(\cdot)$. It is well known [11] that every K-regular prob. measure on E is uniquely determined by its characteristic functional.

Let $\mathfrak{N}$ denote the set of all closed subspaces N of finite codimension of E (*i.e*, the dimension of the quotient space E/N is finite). Then $\mathfrak{N}$ is a directed set under the set-theorectic inclusion $\supset$. For every $N\in\mathfrak{N}$, we denote by P_N the canonical projection mapping $E\to E/N$, and for every $N,M\in\mathfrak{N}$ with $M\subset N$, we denote by P_{NM} the canonical mapping $E/M\to E/N$. A family of measures μ_N on E/N $\{\mu_N\}_{N\in\mathfrak{N}}$ is called a *cylindrical measure* on E if for every $M,N\in\mathfrak{N}$ with $M\subset N$, $P_{NM}(\mu_M)=\mu_N$, Let F be another LCTVS, and $\Phi: E\to F$ be a continuous linear mapping. Then, for a prob. measure μ on E, $\Phi\mu$ is, by definition, the measure $\mu\circ\Phi^{-1}$ on $\mathfrak{B}(F)$. If r is a non-zero real number and $\Phi(x)=rx$, then, we shall use the symbol $T_r\mu$ for $\Phi\mu$; if $r=0$, $T_r\mu=\delta_\theta$, where θ denotes the zero element of E. We note that if $\mu,\nu\in M_K(E)$, then $P_N(\mu*\nu)=P_N\mu*P_N\nu$ and $T_r(\mu*\nu)=T_r\mu*T_r\nu$, for all $N\in\mathfrak{N}$ and $r>0$.

REMARK 2.1. If $\mu\in M_K(E)$, then, for any cofinal subset $\mathfrak{N}_0$ of $\mathfrak{N}$, we can show that $\{P_N\}_{N\in\mathfrak{N}_0}$ becomes a cylindrical measure on E. Moreover, since $\{P_N\}_{N\in\mathfrak{N}_0}$ separates the points of E, it follows from Prokhoroff's theorem [1] that μ is uniquely determined by $\{P_N\}_{N\in\mathfrak{N}_0}$

DEFINITION 2.2. A subset $H\subset M_K(E)$ is called *uniformly tight* (*U.T.*) if for every $\varepsilon>0$, there exists a compact subset K_ε of E such that $\mu(K_\varepsilon^c)<\varepsilon$, for all $\mu\in H$, where $K_\varepsilon^c=E-K_\varepsilon$.

Definition 2.3. A triangular array of measures $\{\lambda_{nj}\}\subset M_K(E)$ $(j=1,2,\cdots,k_n;n=1,2,\cdots)$ is called *uniformly infinitesimal* if for every neighborhood U of θ in E,

$$\lim_n \inf_{1\leq j\leq k_n} \lambda_{nj}(U)=1$$

DEFINITION 2.4. A measure $\mu\in M_K(E)$ is called *infinitely divisible* (*i.d.*) if for each positive integer n, there exists a measure λ_n in $M_K(E)$ such that $\mu=\lambda_n^{*n}$, λ_n convoluted with itself n times.

The following known results are essential, and are stated for future reference.

LEMMA 2.5. (Tortrat's Lemma, [13], p.303). *Let $\mu\in M_K(E)$. Suppose that there exist two cylindrical measures $\{\lambda_N\}_{N\in\mathfrak{N}}$ and $\{\nu_N\}_{N\in\mathfrak{N}}$ on E such that*

$P_N\mu=\lambda_N*\nu_N$, *for all* $N\in\mathfrak{N}$. *If there exists a* $\lambda\in M_K(E)$ *such that* $P_N\lambda=\lambda_N$ *for all* $N\in\mathfrak{N}$, *then, we have a unique* ν *in* $M_K(E)$ *such* $P_N\nu=\nu_N$ *for every* $N\in\mathfrak{N}$, *and* $\mu=\lambda*\nu$.

LEMMA 2.6 ([5], p. 290). *Let* $\mu\in M_K(E)$; *then,* μ *is i.d. if and only if* $P_N\mu$ *is i.d. for all* $N\in\mathfrak{N}$.

LEMMA 2.7. ([5], p. 304). *Let* (λ_{nj}) *be a U.T. triangular array of measures in* $M_K(E)$. *Then the following statements are equivalent*:

(*a*) $\{\lambda_{nj}\}$ *is uniformly infinitesimal.*

(*b*) *for each* $x'\in E'$,

$$\lim_n \sup_{1\leq j\leq k_n}|\hat{\lambda}_{nj}(x')-1|=0. \tag{2.1}$$

3. Characterizations of self-decomposable prob. measures on LCTVS.

Following Loéve [9], we define K-regular self-decomposable prob. measures on LCTVS.

DEFINITION 3.1. Let $\mu\in M_K(E)$; then, μ is called *self-decomposable* if for every $r\in(0,1)$, there exists a measure ν_r in $M_K(E)$ such that

$$\mu=T_r\mu*\nu_r. \tag{3.1}$$

The measure ν_r $(0<r<1)$ will be called the component of μ.

The following proposition shows that a self-decomposable prob. measure and its components are i.d.. Our proof uses the corresponding fact on n-dimensional Euclidean space. This result in a real separable Banach space setting is proved in [8].

PROPOSITION. 3.2. *Let* $\mu\in M_K(E)$ *be self-decomposable. Then,* μ *and its components* ν_r $(0<r<1)$ *are i.d.*.

Proof. Let $N\in\mathfrak{N}$ be fixed; then, by applying P_N on both sides of (3.1), we have, for each $r\in(0,1)$

$$P_N\mu=T_rP_N\mu*P_N\nu_r.$$

This shows that $P_N\mu$ is self-decomposable on E/N. Hence, by [9, p. 323], $P_N\mu$ and $P_N\nu_r$ are *i.d.* on E/N for every $N\in\mathfrak{N}$. Therefore, Lemma 2.6 shows that μ and ν_r are *i.d.* on E.

REMARK 3.3. It follows from the above proposition that if $\mu\in M_K(E)$ is self-decomposable, then $\hat{\mu}(x')\neq 0$ for all $x'\in E$, and hence each $\nu_r(0<r<1)$ is unique.

We now prove the following proposition which will be crucial to the proof of our main theorem of this paper.

PROPOSITION 3.4. *Let $\mu \in M_K(E)$; then, μ is self-decomposable on E if and only if $P_N\mu$ is self-decomposable on E/N for every $N \in \mathfrak{N}$.*

Proof. Since $P_N\mu$ is clearly self-decomposable on E/N for every $N \in \mathfrak{N}$, we need only prove the converse. If μ is degenerate, there is nothing to prove. We thus assume that μ is non-degenerate. Then, we can choose an $N_0 \in \mathfrak{N}$ so that $P_{N_0}\mu$ is non-degenerate. Set $\mathfrak{N}_0 = \{N \in \mathfrak{N} : N \subset N_0\}$ and note that $\mathfrak{N}_0$ is a cofinal subset of $\mathfrak{N}$, and that $T_r\mu$ is uniquely determined by a cylindrical measure $\{P_N(T_r\mu)\}_{N \in \mathfrak{N}_0}$ (Remark 2.1), for every $r \in (0,1)$. Now let $r \in (0,1)$ be fixed; then since $P_N\mu$ is self-decomposable on E/N for every $N \in \mathfrak{N}_0$, we have a measure $\nu_{N,r}$ on E/N such that

$$P_N\mu = T_r P_N\mu * \nu_{N,r} = P_N T_r \mu * \nu_{N,r}. \tag{3.2}$$

Now we assert that $\{\nu_{N,r}\}_{N \in \mathfrak{N}_0}$ is a cylindrical measure on E. To show this, take $M, N \in \mathfrak{N}_0$ with $M \subset N$; then, we would have, from (3.2),

$$\begin{aligned} P_{NM}(P_M\mu) &= P_{NM}(T_r(P_M\mu) * \nu_{M,r} \\ &= (T_r P_{NM} P_M \mu) * (P_{NM}\nu_{M,r}) \\ &= T_r P_N \mu * P_{NM}\nu_{M,r}. \end{aligned}$$

This, along the fact that $P_{NM}(P_M\mu) = P_N\mu$, implies that

$$P_N\mu = T_r P_N \mu * P_{NM}\nu_{M,r}. \tag{3.3}$$

Since the component of $P_N\mu$ is unique (Remark2.2), it follows from (3.2) and (3.3) that we must have $P_{NM}(\nu_{M,r}) = \nu_{N,r}$. Hence $\{\nu_{N,r}\}_{N \in \mathfrak{N}_0}$ is a cylindrical measure on E. Therefore, by Tortrat's lemma (Lemma 2.5) and (3.2), we have a measure ν_r in $M_K(E)$ such that $P_N\nu_r = \nu_{N,r}$, for all $N \in \mathfrak{N}_0$, and $\mu = T_r\mu * \nu_r$. This completes the proof.

The following result is an easy application of Proposition 3.4. This generalizes a result of Kumar [7] proved in real separable Banach space setting.

COROLLARY 3.5. *Let $\mathcal{L}(E)$ denote the set of all self-decomposable measures in $M_K(E)$. Then, $\mathcal{L}(E)$ is a closed subsemigroup of $M_K(E)$.*

Proof. Let $\mu_1, \mu_2 \in M_K(E)$; then, we have, for each $r \in (0,1)$, $\nu_{r,1}, \nu_{r,2} \in M_K(E)$ such that $\mu_1 = T_r\mu_1 * \nu_{r,1}$, and $\mu_2 = T_r\mu_2 * \nu_{r,2}$. So, $\mu_1 * \mu_2 = T_r(\mu_1 * \mu_2) * (\nu_{r,1} * \nu_{r,2})$. Since $\mu_1 * \mu_2 \in M_K(E)$, and $\nu_{r,1} * \nu_{r,2} \in M_K(E)$, it follows that $\mu_1 * \mu_2 \in \mathcal{L}(E)$.

Now we show that $\mathcal{L}(E)$ is closed in $M_K(E)$. To show this, let $\{\mu_\alpha\}$ be a net in $\mathcal{L}(E)$ converging to $\mu \in M_K(E)$. Then, for each $N \in \mathfrak{N}$, the net

$\{P_N\mu_\alpha\}$ in $\mathcal{L}(E/N)$ converges to $P_N\mu \in M_K(E/N)$. Since it is known [6] that $\mathcal{L}(E/N)$ is closed, we have $P_N\mu \in \mathcal{L}(E/N)$, for every $N \in \mathfrak{R}$. Hence, by Proposition 3. 4, μ is self-decomposable on E; thus the proof is complete.

Denote by $\mathcal{L}_1(E)$ the class of those measures μ in $M_K(E)$ for which there exist sequences $\{a_n\} \subset R^+$ and $\{x_n\} \subset E$, and a sequence $\{\mu_n\}$ in $M_K(E)$ such that

(1) $$\lim_n T_{a_n}(\mu_1 * \mu_2 * \cdots * \mu_n) * \delta_{x_n} = \mu$$

and

(2) $\{T_{a_n}\mu_k\}$, $(k=1, 2, \cdots, n; n=1, 2, \cdots)$ is uniformly infinitesimal triangular array.

Further, let $\mathcal{L}_2(E)$ denote the class of those measures $\mu \in M_K(E)$ which satisfies (1) above, and the condition:

(2′) $$a_n \to 0, \text{ and } \frac{a_n}{a_{n+1}} \to 1 \quad \text{as } n \to \infty.$$

We need the following lemma for the proof of the main theorem of this paper. R^n will denote the n-dimensional Euclidean space.

LEMMA 3. 6. *If* $\mu \in \mathcal{L}_2(R^n)$, *then* $\hat{\mu}(x) \neq 0$, *for all* $x \in R^n$.

Proof. Let $\{\mu_n\}$, $\{a_n\}$ and $\{x_n\}$ satisfy (1) and (2′); then, by [9, p. 323], to every $r \in (0, 1)$, there corresponds a subsequence $\{r(n)\}$ of $\{n\}$ such that $n \leq r(n)$ and $\frac{a_{r(n)}}{a_n} \to r$ as $n \to \infty$. If we set $\nu_n = \mu_1 * \mu_2 * \cdots * \mu_n$, then, for each n, we have

$$T_{a_{r(n)}}\nu_{r(n)} * \delta_{x_{r(n)}} = T_{a_{r(n)}/a_n}(T_{a_n}\nu_n * \delta_{x_n}) * \lambda_{r(n)}, \tag{3. 4}$$

where $\lambda_{r(n)} = T_{a_{r(n)}}(\mu_{n+1} * \mu_{n+2} * \cdots \mu_{r(n)}) * \delta_{y_n}$ with $y_n = x_{r(n)} - \frac{a_{r(n)}}{a_n} x_n$. Further, we note that, by Lemma 1[4, p. 2],

$$T_{a_{r(n)}/a_n}(T_{a_n}\nu_n * \delta_{x_n}) \to T_r\mu. \tag{3. 5}$$

Now we suppose that $\hat{\mu}(y_0) = 0$, for some $y_0 \in R^n$; then, we have that the set $A = \{x \in R^n; \hat{\mu}(2x) = 0\}$ is not empty, and it is easy to see that there exists an $x_0 \in A$ such that $||x_0|| = \inf_{x \in A} ||x||$. Therefore, $\widehat{T_r\mu}(2x_0) \neq 0$ for all $r \in (0, 1)$, and it follows from (3. 4) and (3. 5) that $\hat{\lambda}_{r(n)}(2x_0) \to 0$ as $n \to \infty$. Now, we observe an inequality [1, p. 88]

$$|\hat{\lambda}_{r(n)}(2x_0)-\hat{\lambda}_{r(n)}(x_0)|^2\leq 2\{1-\mathrm{Re}\hat{\lambda}_{r(n)}(x_0)\}. \tag{3.6}$$

Moreover, since $\hat{\lambda}_{r(n)}(x_0)\to\frac{\hat{\mu}(x_0)}{\hat{\mu}(rx_0)}$, and $\hat{\lambda}_{r(n)}(2x_0)\to 0$, by letting $n\to\infty$ in (3.6), we have

$$\left|\frac{\hat{\mu}(x_0)}{\hat{\mu}(rx_0)}\right|^2\leq 2\left\{1-\mathrm{Re}\left(\frac{\hat{\mu}(x_0)}{\hat{\mu}(rx_0)}\right)\right\}. \tag{3.7}$$

This leads to a contradiction since, by letting $r\to 1$ in (3.7), we get $1\leq 0$. Hence we complete the proof.

Now we state and prove the main theorem of this paper, which gives two characterizations of K-regular self-decomposable prob. measures in arbitrary LCTVS setting. This extends a result, due to Kumar and Schreiber [8], proved in real separable Banach space setting.

THEOREM 3.7. (Characterizations of self-decomposable prob. measures). *The following statements are equivalent:*

(*a*) $\mu\in\mathcal{L}_1(E)$.

(*b*) $\mu\in\mathcal{L}_2(E)$.

(*c*) μ *is* (*K-regular*) *self-decomposable.*

Proof. (a) implies (b): Let $\mu\in\mathcal{L}_1(E)$, and let $\{\mu_n\}$, $\{a_n\}$ and $\{x_n\}$ satisfy (1) and (2). Choose an $N_0\in\mathfrak{N}$ such that $P_{N_0}\mu$ is non-degenerate (note that μ is non-degenerate). Then, since $\{T_{a_n}P_{N_0}\mu_k\}$ is clearly uniformly infinitesimal on E/N_0, it follows that $P_{N_0}\mu\in\mathcal{L}_1(E/N_0)$. Hence, by the well-known classical result in [9, p. 319], we have $a_n\to 0$ and $\frac{a_n}{a_{n+1}}\to 1$.

(b) implies (c): Let $\mu\in\mathcal{L}_2(E)$, and let $\{\mu_n\}$, $\{a_n\}$ and $\{x_n\}$ satisfy (1) and (2′). Choose an $N_0\in\mathfrak{N}$ so that $P_{N_0}\mu$ is non-degenerate, and we define $\mathfrak{N}_0=\{N\in\mathfrak{N}: N\subset N_0\}$, Then, clearly, $P_N\mu\in\mathcal{L}_2(E/N)$ for all $N\in\mathfrak{N}_0$; and, therefore, by Lemma 3.6, $\widehat{P_N\mu}(\cdot)$ has no zeros. Now let $N\in\mathfrak{N}_0$ and $r\in(0,1)$ be fixed, and let $\{r(n)\}$ be a subsequence of $\{n\}$ as chosen in the proof of Lemma 3.6. If we set $\nu_n=P_N(\mu_1*\mu_2*\cdots*\mu_n)$, then, for every n, we have

$$T_{a_{r(n)}}\nu_{r_n}*\delta_{P_N(x_{r(n)})}=T_{a_{r(n)}/a_n}(T_{a_n}\nu_n*\delta_{P_N(x_n)})*\lambda_{r(n)}, \tag{3.8}$$

where $\lambda_{r(n)}=T_{a_{r(n)}}(P_N\mu_{n+1}*P_N\mu_{n+2}*\cdots*P_N\mu_{rn})*\delta_{y_n}$ with $y_n=P_N(x_{r(n)})-\frac{a_{r(n)}}{a_n}P_N(x_n)$. Since $T_{a_n}\nu_n*\delta_{P_N(x_n)}\to P_N\mu$, $\frac{a_{r(n)}}{a_n}\to r$ and $\widehat{P_N\mu}(x)\neq 0$ for all n, it follows from (3.8) that

$$\hat{\lambda}_{r(n)}(x) \to \frac{\widehat{P_N\mu}(x)}{\widehat{P_N\mu}(rx)} \overset{\text{def}}{=} \widehat{\lambda_{N,r}}(x),$$

for all $x \in R^n$, where $\lambda_{N,r} \in M_K(E/N)$ with $\hat{\lambda}_{N,r}(x) = \dfrac{\widehat{P_N\mu}(x)}{\widehat{P_N\mu}(rx)}$ (Lévy continuity theorem). Hence we have $P_N\mu = T_r(P_N\mu) * \lambda_{N,r}$. This shows that $P_N\mu$ is self-decomposable on E/N. Since N is arbitrary in $\mathfrak{N}_0$, it follows from Proposition 3.4 that μ is self-decomposable.

(c) implies (a): Let us take a sequence $\{r_k\}$ in $(0, 1)$ defined by $r_k = \dfrac{k-1}{k}$ $(k=2, 3, \cdots)$. Then, since μ is self-decomposable, we have, for each $k(=2, 3, \cdots)$, $\nu_{r_k} \in M_K(E)$ such that

$$\mu = T_{r_k}\mu * \nu_{r_k} \tag{3.9}$$

Moreover, from Remark 2.2 and (3.9), we have

$$\hat{\nu}_{r_k}(x') = \frac{\hat{\mu}(x')}{\hat{\mu}(r_k x')}, \tag{3.10}$$

for all $x' \in E'$. Now let $\lambda_1 = \mu$, and $\lambda_k = T_k \nu_{r_k}$, for $k = 2, 3, \cdots$. Then, $\{\lambda_k\}_{k=1}^{\infty} \subset M_K(E)$, and, from (3.10), we have, for each $x' \in E'$,

$$\hat{\lambda}_k(x') = \frac{\hat{\mu}(kx')}{\hat{\mu}((k-1)x')}, \quad k = 1, 2, \cdots.$$

From the identity

$$\hat{\mu}(nx') = \hat{\mu}(x') \cdot \frac{\hat{\mu}(2x')}{\hat{\mu}(x')} \cdots \frac{\hat{\mu}(nx')}{\hat{\mu}((n-1)x')},$$

we obtain, for every n,

$$\mu = T_{\frac{1}{n}}(\lambda_1 * \lambda_2 * \cdots * \lambda_n). \tag{3.11}$$

Now we will show that $\{T_{\frac{1}{n}}\lambda_k\}$ $(k=1, 2, \cdots, n;\ n=1, 2, \cdots)$ is uniformly infiniteimal. To show this, from Lemma 2.7, we need to show that $\{T_{\frac{1}{n}}\lambda_k\}$ is $U.T.$, and satisfies (2.1). Since μ is K-regular, it is easily shown that $\{T_r\mu\}_{r \in (0,1)}$ is $U.T.$, and hence, by Theorem 2.1 [10, p. 58], $\{\nu_r\}_{r \in (0,1)}$ is also $U.T.$, Therefore, for given $\varepsilon > 0$, there exists a compact subset K_ε such that

$$T_r\mu(K_\varepsilon) \geq 1 - \varepsilon, \text{ and } \nu_r(K_\varepsilon) \geq 1 - \varepsilon,$$

for all $r \in (0, 1)$. By defining a compact set $K = \bigcup_{t \in [0,1]} tK_\varepsilon$, we infer that, for every n,

$$T_{\frac{1}{n}}\lambda_k(K)=\begin{cases}T_{\frac{1}{n}}\mu(K)\geq 1-\varepsilon, & \text{for } k=1,\\ T_{\frac{k}{n}}\nu_{r_k}(K)\geq \nu_{r_k}(K)\geq 1-\varepsilon, & \text{for } k=2,3,\cdots n,\end{cases}$$

which implies that $\{T_{\frac{1}{n}}\lambda_k\}$ is $U.T.$. Now we prove that $\{T_{\frac{1}{n}}\lambda_k\}$ satisfies (2.1). We first observe that, for every n, we have the inequality,

$$\begin{aligned}\sup_{1\leq k\leq n}|\widehat{T_{\frac{1}{n}}\lambda_k}(x')-1|^2&=\sup_{1\leq k\leq n}\left|\frac{\hat{\mu}\left(\frac{k}{n}x'\right)}{\hat{\mu}\left(\frac{k-1}{n}x'\right)}-1\right|^2\\&=\sup_{1\leq k\leq n}\left|\frac{\hat{\mu}\left(\frac{k}{n}x'\right)-\hat{\mu}\left(\frac{k-1}{n}x'\right)}{\hat{\mu}\left(\frac{k-1}{n}x'\right)}\right|^2\\&\leq\frac{2\left|1-\mathrm{Re}\left(\hat{\mu}\left(\frac{x'}{n}\right)\right)\right|}{\inf_{1\leq k\leq n}\left|\hat{\mu}\left(\frac{k-1}{n}x'\right)\right|^2}\end{aligned}\qquad(3.12)$$

Now we consider the real valued function $\phi:[0,1]\to R$ defined by $\phi(a)=|\hat{\mu}(ax')|^2$ (x' is fixed). Then, clearly, ϕ is continuous on the compact set $[0,1]$. Hence there exists $a_0\in[0,1]$ such that $\inf_{a\in[0,1]}\phi(a)=|\mu(a_0x')|^2$, Thus, for each n,

$$\begin{aligned}\inf_{1\leq k\leq n}|\hat{\mu}\left(\frac{k-1}{n}x'\right)|^2&\geq\inf_{a\in[0,1]}|\hat{\mu}(ax')|^2\\&=|\hat{\mu}(a_0x')|^2>0\quad\text{(Remark 2.2)}.\end{aligned}$$

Moreover, since $\lim_n|1-\mathrm{Re}(\hat{\mu}(\frac{x'}{n}))|=0$, it follows from (3.12) that

$$\lim_n\sup_{1\leq k\leq n}\left|\widehat{T_{\frac{1}{n}}\lambda_k}(x')-1\right|^2=0.$$

Take $a_n=\frac{1}{n}$, $x_n=\theta$ and $\mu_n=\lambda_n$ for $n=1,2,\cdots$, then (3.11) completes the proof.

DEFINITION 3.8. ([4], p. 5) Let $\mu\in M_K(E)$; then μ is called *stable* if for $a,b\in R^+$, there exist $c\in R^+$ and $x\in E$ such that

$$T_a\mu*T_b\mu=T_c\mu*\delta_x.$$

COROLLARY 3.9. *If a measure μ in $M_K(E)$ is stable, then μ is self-decomposable.*

Proof. If μ is degenerate, then, clearly it is self-decomposable. Thus we assume that μ is non-degenerate. By Theorem 2 of [4], there exist a $\nu\in M_K(E)$ and sequences $\{b_n\}$ in R^+ and $\{x_n\}$ in E such that

$$\lim_n (T_{a_n}\nu^{*n} * \delta_{x_n}) = \mu.$$

Since μ is non-degenerate, one can show that $\lim_n a_n=0$. Hence $\{T_{a_n}\nu\}$ is a uniformly infinitesimal triangular array of measures in $M_K(E)$. Thus $\mu\in\mathcal{L}_1(E)$; therefore, by Theorem 3.7, μ is self-decomposable.

4. The support of symmetic self-decomposable prob. measures on LCTVS.

Let μ be a prob. measure on E; then the smallest closed set with full μ-measure is called the *support* of μ. If $\mu\in M_K(E)$, then the support of μ always exists [3]; S_μ will denote the support of μ.

For the proof of the theorem in this section, we will need the following two results.

LEMMA 4.1. ([3], p. 37) *Let μ and ν be K-regular prob. measures on LCTVS E. Then $S_{\mu*\nu}=\overline{S_\mu+S_\nu}$, where $\overline{S_\mu+S_\nu}$ is the closure of $S_\mu+S_\nu$ in E.*

LEMMA 4.2. (Rajput [12]) *If $\mu\in M_K(E)$ is symmetric i.d., then S_μ is a (closed) subgroup (under addition) of E.*

Now we are ready to state and prove the following theorem.

THEOREM 4.3. *If $\mu\in M_K(E)$ is symmetric self-decomposable, then* S_μ *is a subspace of E.*

Proof. Since μ is symmetric and *i.d.* (Proposition 3.2), it follows from Lemma 4.2 that S_μ is a subgroup of E. Thus, the proof will be complete if we can show that $aS_\mu\subset S_\mu$ for all $a\in R$. We first show that the component ν_r is symmetric for every $r\in(0,1)$. Since μ is symmetric, we have, for every $r\in(0,1)$,

$$\mu=T_{-1}\mu=T_{-1}(T_a\mu * \nu_r)=T_a\mu * T_{-1}\nu_r.$$

By the uniqueness of the component (Remark 3.3), we must have $T_{-1}\nu_r=\nu_r$, which implies that ν_r is symmetric. Further, since, by Proposition 3.2, ν_r is *i.d.*, it follows from Lemma 4.2 that S_{ν_r} is a subgroup of E for every $r\in(0,1)$. Noting the fact $S_{T_a\mu}=aS_\mu$, $a>0$, we obtain, by Lemma 4.1,

$$S_\mu=\overline{rS_\mu+S_{\nu_r}},$$

for every $r\in(0,1)$. Therefore, since $\theta\in S_{\nu_r}$ and $S_\mu=-S_\mu$, we have $S_\mu\supset rS_\mu$ for $|r|<1$. Now let a be any real number; then $a=n+r$ for some integer n and a real number $|r|<1$, and we have

$$aS_\mu\subset nS_\mu+rS_\mu\subset S_\mu+S_\mu=S_\mu.$$

Hence we complete the proof.

COROLLARY 4.4. ([12]) *If $\mu\in M_K(E)$ is symmetric stable, then S_μ is a subspace of E.*

Proof. The proof follows from Theorem 4.3 and Corollary 3.9.

References

1. N. Bourbaki, *Eléments de Mathématique*, Livre VI Intégration, Chapitre IX, Paris, France: Hermann, 1969.
2. I. Csiszár, *Some problems concerning measures on topological spaces and convolutions of measures on topological groups*, Les Probabilitiés sur les Structures Algébriques (Colloques Internat. de CNRS, Paris, 1970), Clemont-Ferrand, 75-79.
3. ______, *On the weak* continuity of convolution in a convolution algebra over an arbitrary group*, Studia Sci. Math. Hung. **6** (1971), 27-40.
4. I. Csiszár and B.S. Rajput, *A convergence of types theorems probability measures on TVS with applications to stable laws*, Z. Wahrscheinlichkeitstheorie verw. Gebeite **36** (1976), 1-7.
5. E. Dettweiler, *Grenzwertsätze für Wahrscheinlichkeitmasse auf Badrikianschen Räumen*, Z. Wahrscheinlichkeitstheorie verw. Gebeite **34** (1976), 285-311.
6. L. Kubik, *A characterization of the class L of probality distributions*, Studia Math. **21** (1972) 245-252.
7. A. Kumar, *A note on the convergence of stable and class L probability measures on Banach spaces*, Ann. Probability **1** (1973), 716-718.
8. A. Kumar and B. Schreiber, *Self-decomposable probability measures on Banach spaces*, *Studia* Math. **53** (1975), 55-71.
9. M. Loève, *Probability Theory*, New York: Van Nostrand, 1955.
10. K. S. Parthasarathy, *Probability Measures on Metric Spaces*, New York: Academic Press, 1967.
11. Y. Prokhoroff, *The method of characteristic functional*, Proc. of IV Berkeley Symposium, Vol. II, University of California Press, Berkeley, 1961, 403-419.
12. B. Rajput, *On the support of symmetric infinitely divisible and stable probability measures on LCTVS*, Proc. Amer. Math. Soc. **66** (1977), 331-334.
13. A Tortrat, *Structure des lois indéfiniment divisibles dans un espace vectoriel topologique*, Lecture Notes in Math. **31** (1967), New York: Springer-Verlag, 299-327.

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