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REGULAR TRANSFORMATION GROUPS

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In this paper, we will show some relations between regular minimal sets and minmial proximal transformation groups, and a necessary and sufficient condition for a transformation group to be proximal. By using these results, we will develop the concept of regular transformation groups as a generalization of that of regular minimal sets and proximal transformation groups, and show that the proximal relation of a transformation group is an equivalence relation if and only if its enveloping semigroup is a regular transformation group. Furthermore, we will give a sufficient condition for every orbit closure of a transformation group to be regular.

In this paper, let T be an arbitrary, but fixed, topological group and we consider the (right) transformation group (X, T) with a compact Hausdorff phase space X. A closed nonempty subset A of X is said to be a *minimal set* if, for every $x \in A$, the orbit xT is a dense subset of A. A point whose orbit closure is a minimal set is called an *almost periodic point*. If X is itself minimal, we say it is a *minimal transformation group*. The points x and y of X in the transformation group (X, T) are called *proximal* provided that for each neighborhood W of the diagonal $\varDelta$ of $X \times X$, there exists a $t \in T$ such that $(xt, yt) \in W$. We denote $P(X, T) = \{(x, y) \in X \times X \mid x$ and y are proximal$\}$ which is called the *proximal relation* on (X, T). The transformation group (X, T) is said to be *proximal* if every two points of X are proximal.

If (Y, T) is also a transformation group, a *homomorphism* from (X, T) to (Y, T) is a continuous map $\phi : X \to Y$ such that $\phi(xt) = \phi(x)t$ $(x \in X, t \in T)$. If (Y, T) is minimal, ϕ is always onto. Especially, a homomorphism from (X, T) into itself (not necessarily onto) is called an *endomorphism* of (X, T), and if ϕ is bijective then ϕ is called an *automorphism* of (X, T).

As is customary, let X^X denote the set of all functions from X to X, provided with the topology of pointwise convergence, and consider T as the subset $\{t : x \to xt, t \in T\}$ of X^X. The *enveloping semigroup* $E(X)$ of the transformation group (X, T) is the closure of T in X^X. Then $E(X)$ is a

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compact Hausdorff space and we may consider $(E(X), T)$ as a transformation group, whose phase space $E(X)$ admits a semigroup structure. The *minimal right ideals* I of $E(X)$ (that is, the nonempty subsets I of $E(X)$ such that $IE(X)\subset I$, which contains no proper nonempty subsets with the same property) concide with the minimal sets in the transformation group $(E(X), T)$ (see 3.4[4]).

In [4], Ellis showed that if (X, T) is a transformation group and $E(X)$ its enveloping semigroup, then for each $x\in X$, the map $\theta_x: p\rightarrow xp=p(x)$ of $E(X)$ into X is a homomorphism, and its image is just the orbit closure $\overline{xT}$ of x, i.e., $\theta_x: (E(X), T)\rightarrow (\overline{xT}, T)$.

He also showed that if $\phi: (X, T)\rightarrow (Y, T)$ is an epimorphism, then there exists a unique epimorphism $\theta: (E(X), T)\rightarrow (E(Y), T)$ such that the diagram

$$\begin{array}{ccc} E(X) & \xrightarrow{\theta} & E(Y) \\ \downarrow{\scriptstyle \theta_x} & & \downarrow{\scriptstyle \theta_{\phi(x)}} \\ X & \xrightarrow{\phi} & Y \end{array}$$

commutes $(x\in X)$ (see Proposition 3.8 in [4]).

Henceforth, θ and θ_x will denote the above homomorphisms.

In [1], Auslander showed that if (X, T) is minimal, then the following are pairwise equivalent:

(a) If I is a minimal right ideal contained in the enveloping semigroup $E(X)$ of (X, T), then (X, T) and (I, T) are isomorphic.

(b) (X, T) is isomorphic with (I, T), where I is a minimal right ideal in the enveloping semigroup of some transformation group (Z, T).

(c) If $x, y\in X$, then there is an endomorphism ϕ of (X, T) such that $\phi(x)$ and y are proximal.

(d) If (x, y) is an almost periodic point of $(X\times X, T)$, then there exists an endomorphism ϕ of (X, T) such that $\phi(x)=y$.

A minimal set which satisfies any one of the properties (a) through (d) will be called *regular minimal* [1].

In [7], Glasner studied the proximal transformation groups and proved the following theorem:

THEOREM 1. (Glasner) (1) *The only endomorphism of a minimal proximal transformation group is the identity.*

(2) *A transformation group (X, T) is proximal if and only if any minimal set in $(X\times X, T)$ is contained in the diagonal.*

(3) *A proximal transformation group contains exactly one minimal set.*

(4) *If* (X, T) *is a minimal proximal transformation group, then* $(E(X), T)$ *is proximal,* $E(X)$ *contains a unique minimal ideal* I, *and* (I, T) *is isomorphic to* (X, T).

The following theorem will show a relation between regular minimal sets and minimal proximal transformation groups and simultaneously the converse of (1) in Theorem 1.

THEOREM 2. *Let* (X, T) *be a minimal transformation group. Then the following are equivalent*:

(1) (X, T) *is proximal.*

(2) (X, T) *is regular minimal and the only endomorphism of* (X, T) *is the identity.*

Proof: (1) implies (2). By Auslander, (X, T) is regular if and only if for each x, y in X there exists an endomorphism ϕ of (X, T) such that $\phi(x)$ and y are proximal.

Let x and y be any two elements of (X, T). Then for the identity i we have $(i(x), y) = (x, y) \in P(X, T)$. Since the identity is an endomorphism of (X, T), it follows that (X, T) is regular minimal. The remaining part was proved by Glasner.

(2) implies (1). Let x and y be any two points of (X, T). Since (X, T) is regular minimal, there exists an endomorphism ϕ of (X, T) such that $(\phi(x), y) \in P(X, T)$. But the only endomorphism is the identity. Therefore ϕ is the identity. Thus $(x, y) = (i(x), y) = (\phi(x), y) \in P(X, T)$.

REMARK 3. Every proximal transformation group contains exactly one regular minimal set M and every endomorphism of the proximal transformation group is the identity on M.

Proof: By (3) of Theorem 1 and Theorem 2.

In [1], Auslander proved the following theorem:

THEOREM 4. *Let* (X, T) *be a minimal transformation group and* (M, T) *be any minimal set in* $(X \times X, T)$. *Then* (X, T) *is regular minimal if and only if* (X, T) *is isomorphic to* (M, T) *and every endomorphism of* (X, T) *is an automorphism.*

For the minimal proximal transformation groups, we have the following:

THEOREM 5. *Let* (X, T) *be a minimal transformation group and* (M, T) *be any minimal set in* $(X \times X, T)$. *Then* (X, T) *is proximal if and only if* (X, T) *is isomorphic to* (M, T) *and the only endomorphism of* (X, T) *is the identity.*

Proof: From Theorem 2 and Theorem 4, we can obtain Theorem 5.

REMARK 6. By Theorem 4 and Theorem 5, a regular minimal set can be considered as a generalization of a minimal proximal transformation group.

LEMMA 7. *Let ϕ be an endomorphism of (X, T) (not necessarily onto),* and let $Y=\phi(X)$ *and* $\theta:(E(X), T)\rightarrow(E(Y), T)$. *Then* $\theta(p)$ *is identical with* p *on* Y.

Proof: Let p be any element of $E(X)$. Then there exists a net (t_α) in T such that $t_\alpha\rightarrow p$ in X^X. This is equivalent to $xt_\alpha\rightarrow xp$ in X $(x\in X)$. Now $\theta(p)\in E(Y)$. This means that $t_\alpha\rightarrow\theta(p)$ in Y^Y. This is equivalent to $yt_\alpha\rightarrow y\theta(p)$ in Y $(y\in Y)$. Since $Y\subset X$, it follows that $yt_\alpha\rightarrow yp$ in X for each $y\in Y$. Since Y is a closed invariant subset of X and $yt_\alpha\rightarrow yp$, we have $yp\in Y$. Therefore, p belongs to Y^Y and $yp=\lim yt_\alpha=y\theta(p)$ for each $y\in Y$. Thus we have $\theta(p)=p$ on Y.

Ellis' result [5, (5) of Lemma 4] is a corollary to the above lemma:

COROLLARY 8. *Let ϕ be an endomorphism from (X, T) onto (X, T). Then θ is the idenity map of $E(X)$ onto $E(X)$.*

We give another necessary and sufficient condition for a transformation group to be proximal.

THEOREM 9. *A transformation group (X, T) is proximal if and only if for each $x, y\in X$, there exists an endomorphism ϕ of (X, T) such that*

(i) $(\phi(x), y)\in P(X, T)$, *and*

(ii) ϕ *is the identity on any minimal subset of* (X, T).

Proof: Only if: If we take the identity as an endomorphism, it satisfies (i) and (ii).

If: By Theorem 1 (2), it is sufficient to show that any minimal set in $(X\times X, T)$ is contained in the diagonal. Let M be a minimal set in $(X\times X, T)$ and $(x, y)\in M$. Then there exists an endomorphism ϕ of (X, T) such that $(\phi(x), y)\in P(X, T)$ and ϕ is the identity on any minimal subset of (X, T). Since $(y', y)\in P(X, T)$, where $y'=\phi(x)$, there is a minimal right ideal I in $E(X)$ such that $yp=y'p$ for any $p\in I$. Since (x, y) is an almost periodic point of $(X\times X, T)$, there exists an idempotent u of I such that $(x, y)u=(x, y)$. Then we have $xu=x$ and $yu=y$. Hence $y'u=yu=y$. Therefore, we get $\phi(x)=\phi(xu)=\phi(x)u=y'u=y$ by Lemma 7.

Let $\pi: X\times X\rightarrow X$ be a projection. Since M is a minimal set in $(X\times X, T)$,

$\pi(M)$ is also a minimal subset of (X, T) and $x \in \pi(M)$. Therefore, ϕ is the identity on $\pi(M)$ by (ii). Thus we have $y=\phi(x)=x$. Therefore, M is contained in the diagonal of $(X \times X, T)$.

If we consider Remark 6 and Theorem 9, we can define a regular transformation group as follows:

DEFINITION 10. A transformation group (X, T) (not necessarily minimal) is *regular* if for each $x, y \in X$, there exists an endomorphism ϕ of (X, T) such that

(i) $(\phi(x), y) \in P(X, T)$, and

(ii) ϕ is an endomorphism (hence, an automorphism) on any minimal subset of (X, T).

The points x and y are called *regular* if there exists an endomorphism satisfying (i) and (ii).

REMARK 11. (1) If (X, T) is regular minimal, then (X, T) is regular.
(2) Every proximal transformation group is regular by Theorem 9.
(3) The converses of (1) and (2) do not hold (see Example 18).

REMARK 12. From Theorem 1 (4), it follows that a minimal transformation group (X, T) is proximal if and only if $(E(X), T)$ is proximal.

Auslander [1] showed that minimal right ideals of enveloping semigroups are regular minimal. Therefore, (X, T) is distal if and only if $(E(X), T)$ is regular minimal.

Let (X, T) be a minimal set. Then we have the following diagram:

$$\begin{array}{ccc}
P(X, T)=X\times X & \Longleftrightarrow & (E(X), T):\ \text{proximal} \\
\Big\downarrow \not\Big\uparrow & & \not\Big\uparrow \Big\downarrow \\
P(X, T)=\text{equivalence relation (Thm 17)} & \Longleftrightarrow & (E(X), T):\ \text{regular} \\
\Big\uparrow \not\Big\downarrow & & \not\Big\downarrow \Big\uparrow \\
P(X, T)=\Delta\,(\text{diagonal}) & \Longleftrightarrow & (E(X), T):\ \text{regular minimal}
\end{array}$$

THEOREM 13. *A transformation group (X, T) is regular if and only if for each almost periodic point (x, y) in $(X \times X, T)$, there exists an endomorphism ϕ of (X, T) such that $\phi(x)=y$ and ϕ is also an endomorphism on any minimal subset of (X, T).*

Proof: If: Let x and y be any two elements of X. Then $\overline{(x, y)T}$ contains a minimal set M. Then there exist a net (t_α) in T and $(z_1, z_2) \in M$ such that $(x, y)t_\alpha \to (z_1, z_2)$.

Since (z_1, z_2) is an almost periodic point of $(X\times X, T)$, there exists an endomorphism ϕ of (X, T) with $\phi(z_1)=z_2$ which is also an endomorphism on any minimal subset of (X, T). Since $z_1=\lim xt_\alpha$ and $z_2=\lim yt_\alpha$, we get $\lim \phi(x)t_\alpha=\phi(\lim xt_\alpha)=\phi(z_1)=z_2=\lim yt_\alpha$. Therefore, we have $(\phi(x), y)\in P(X, T)$.

Only if: Let (x, y) be an almost periodic point of $(X\times X, T)$. Then there exists an endomorphism ϕ of (X, T) such that $(\phi(x), y)\in P(X, T)$ and ϕ is also an endomorphism on any minimal subset of (X, T). Since $(\phi(x), y) \in P(X, T)$, we have a minimal right ideal I of $E(X)$ such that $\phi(x)p=yp$, $p\in I$. Since (x, y) is an almost periodic point, there exists an idempotent u in I such that $(x, y)u=(x, y)$. Now $y=yu=\phi(x)u$. By Lemma 7, we obtain $\phi(x)\theta(u)=\phi(x)u$, where $\theta: E(X)\to E(\phi(X))$. Therefore, we have $\phi(x)=\phi(xu)=\phi(x)\theta(u)=\phi(x)u=y$.

Lemma 14. *If (X, T) is a regular transformation group, then (X, T) contains exactly one minimal set.*

Proof: Suppose there exist two minimal subsets M_1, M_2 of (X, T). Let $x_1\in M_1$ and $x_2\in M_2$. Then there exists an endomorphism ϕ of (X, T) with $(\phi(x_1), x_2)\in P(X, T)$ which is also an endomorphism on any minimal subset of (X, T). Since $x_1\in M_1$, $\phi(x_1)$ belongs to M_1. Now, since $(\phi(x_1), x_2)\in P(X, T)$, there exists a net (t_α) in T such that $\lim \phi(x_1)t_\alpha=\lim x_2t_\alpha$. Therefore, we obtain a nonempty common part of M_1 and M_2. Thus we get $M_1=M_2$.

Theorem 1 (3) is a corollary to the above lemma:

Corollary 15. *Every proximal transformation group contains exactly one minimal set.*

Lemma 16. *Let (X, T) be a transformation group. The transformation group $(E(X), T)$ is regular if and only if $E(X)$ contains exactly one minimal right ideal.*

Proof: Only If: Since the minimal subsets of $(E(X), T)$ coincide with the minimal right ideals of $E(X)$, this part is clear by Lemma 14.

If: Let p and q be any two elements of $E(X)$. Since $\overline{(p, q)T}$ contains a minimal subset M in $(E(X)\times E(X), T)$, there exist a net (t_α) in T and a point (z_1, z_2) in M such that $\lim (p, q)t_\alpha=(z_1, z_2)$.

Now the minimal right ideal I of $(E(X), T)$ is a regular minimal set and $I=\pi(M)$, where π $(E(X)\times E(X), T)\to (E(X), T)$: is a projection. Since there is only one minimal set in $(E(X), T)$, we have $z_1, z_2\in\pi(M)=I$. Since (z_1, z_2) is an almost periodic point of $(I\times I, T)$, there exists an endomorphism ϕ of (I, T) such that $\phi(z_1)=z_2$.

By Theorem 3 in [2], there exists $r\in I$ such that $\phi=L_r$. We may assume that L_r is an endomorphism of $(E(X), T)$ such that $L_r=\phi$ on I. Therefore, we have $\lim qt_\alpha=z_2=\phi(z_1)=L_r(z_1)=L_r(\lim\ pt_\alpha)=\lim\ L_r(p)t_\alpha$. Thus there exists an endomorphism L_r of $(E(X), T)$ such that $(L_r(p), q)\in P(E(X), T)$ Therefore, $(E(X), T)$ is a regular transformation group.

THEOREM 17. *Let (X, T) be a transformation group. Then $P(X, T)$ is an equivalence relation if and only if the transformation group $(E(X), T)$ is regular.*

Proof: Use Lemma 16 and Proposition 5.16 in (4).

EXAMPLE 18. (1) We show that there exists a regular transformation group which is neither regular minimal nor proximal.

By Remark 12, it is sufficient to show that there exists a minimal transformation group (X, T) with an equivalence relation $P(X, T)$ such that $P(X, T)$ is neither $X\times X$ nor the diagonal of $X\times X$.

The following example is from [4].

Let Y be a circle. For $a, b\in Y$ let (a, b) be the open arc from a to b traversed in a counterclockwise direction, and $[a, b)=\{a\}\cup(a, b)$.

Let $X=(Y\times 1)\cup(Y\times 2)$, i.e., two copies of Y, and let $\tau: X\to X$ be defined by $\tau(a, 1)=(a, 2)$ and $\tau(a, 2)=(a, 1)$ $(a\in Y)$.

Make X into a topological space by defining a typical neighborhood of the point $(a, 1)$ to be the set $[a, b)\times 1\cup(a, b)\times 2$ with $b\neq a$ and a typical neighborhood of $(a, 2)$ to be the set $(b, a)\times 1\cup(b, a]\times 2$ with $b\neq a$. Then X is compact Hausdorff. Now let φ be a rotation through one radian on Y and $\Psi: X\to X$ such that $\Psi(a, i)=(\varphi(a), i)$, $i=1, 2$. Then (X, T) is minimal and $P(X, T)=\{(x, x)\mid x\in X\}\cup\{x, \tau x)\mid x\in X\}$ is an equivalence relation which is neither $X\times X$ nor the diagonal of $X\times X$, where $T=\{\Psi^n\mid n\in\mathbf{Z}\}$.

(2) If (X, T) is a nontrivial locally almost periodic minimal transformation group which is not (uniformly) almost periodic, then $(E(X), T)$ is a regular transformation group which is neither proximal nor regular minimal (see Corollary 2.1 in [6]).

Auslander [3] showed that every almost periodic minimal set with an abelian acting group is regular minimal. In fact, these transformation groups satisfy the following conditions: (1) (X, T) is minimal, (2) $P(X, T)$ is the diagonal of $X\times X$, (3) T is abelian, and (4) there is a minimal right ideal I of $E(X)$ all of whose elements are continuous.

If we drop the minimality of (X, T) and substitute that $P(X, T)$ is an equivalence relation instead of that $P(X, T)$ is the diagonal of $X\times X$, we have the following:

THEOREM 19. *Let* (X, T) *be a transformation group such that* $P(X, T)$ *is an equivalence relation and* T *is abelian. Suppose there is a minimal right ideal of* $E(X)$ *all of whose elements are continuous. Then every orbit closure is regular.*

Proof: If we consider $\theta_x : (E(X), T) \to (\overline{xT}, T)$, then θ_x is an epimorphism. Since $P(X, T)$ is an equivalence relation, we have that $(E(X), T)$ is regular by Theorem 17.

Let z_1 and z_2 be any two points of $\overline{xT}$. Then there exist p and q in $E(X)$ such that $\theta_x(p)=z_1, \theta_x(q)=z_2$. Since $(E(X), T)$ is regular, there exists an endomorphism ϕ of $(E(X), T)$ such that $(\phi(p), q) \in P(E(X), T)$. In fact, we may assume $\phi=L_r$, where r is an element of $I \subset E(X)$, (see the proof of Lemma 16).

Since θ_x is an epimorphism, we get $(\theta_x L_r(p), \theta_x(q)) \in P(\overline{xT}, T)$. Since T is abelian and r is continuous, we get $\theta_x L_r(p)=xrp=xpr=z_1 r$.

It is sufficient to show that r is an endomorphism of $(\overline{xT}, T)$ satisfying the conditions for regular transformation groups. Let z be any element of $\overline{xT}$. Then we get $z=\lim xt_\alpha$ and $zr=(\lim xt_\alpha)r=\lim((xt_\alpha)r)=\lim(xr)t_\alpha \in \overline{xT}$. Therefore, r is an endomorphism of $(\overline{xT}, T)$ since T is abelian and r is continuous.

Since $(E(X), T)$ is regular, $E(X)$ has only one minimal set. Thus $(\overline{xT}, T)$ has only one minimal set since $\overline{xT}=\theta_x(E(X))$. Let M be the minimal subest of $\overline{xT}$. Then $Mr=M$. Therefore, for any z_1, z_2 in $(\overline{xT}, T)$, there exists an endomorphism r of $(\overline{xT}, T)$ such that $(z_1 r, z_2) \in P(\overline{xT}, T)$ and r is also an endomorphism of any minimal subset of $\overline{xT}$. Thus $(\overline{xT}, T)$ is regular.

COROLLARY 20. *Let* (X, T) *be almost periodic and* T *be abelian. Then every orbit closure is regular minimal. In particular, almost periodic minimal sets with abelian acting groups are regular minimal..*

We will show the existence of a transformation group satisfying the conditions of Theorem 19 which is not (uniformly) almost periodic; that is, $P(X, T)$ is an equivalence relation on X, T is abelian and there exists a minimal right ideal of $E(X)$ all of whose elements are continuous.

EXAMPLE 21. Let X be the set of complex numbers z such that $2 \leqslant |z| \leqslant 3$. Let $\varphi(r \exp(2\pi it))=r \exp(2\pi it^r)$ $(2 \leqslant r \leqslant 3,\ 0 \leqslant t \leqslant 1)$ and $T=\{\varphi^n \mid n \in \mathbf{Z}\}$. Then (X, T) is a transformation group and satisfies the following:

(1) $P(X, T)=\{(u, v) \mid |u|=|v| \text{ and } u, v \in X\}$ is an equivalence relation on X.

(2) T is abelian, and

(3) If Ψ is a function from X to X defined by $\Psi(r\ \exp(2\pi it))=r$, then Ψ belongs to $E(X)$ since $\varphi^n \to \Psi$ in X^X. Now Ψ is continuous, because for any basic open neighborhood $B_l(r)$ of r,

$$\Psi^{-1}(B_l(r)) = \{z \mid r-l < |z| < r+l, z \in X\}$$

is open in X. Let $I=\{\Psi\}$. Then I is closed since $E(X)$ is Hausdorff. For each n, we have $z\Psi\varphi^n = |z|\varphi^n = |z| = z\Psi$ for any $z \in X$. Thus I is invariant. Since I has only one element, I is minimal. Therefore, I is a minimal right ideal of $E(X)$ all of whose elements are continuous.

But this is not (uniformly) an almost periodic transformation group.

REMARK 22. If we consider Remark 12, we have the following: Let (X, T) be a minimal set. Then $P(X, T)$ is an equivalence relation which is neither $X \times X$ nor the diagonal if and only if $(E(X), T)$ is regular and is neither proximal nor regular minimal.

In this case, the assumption of the minimality of X is necessary. Becau-se, by the above example, there exists a transfomation group which is not proximal but its enveloping semigroup is proximal. (Since $P(X, T)$ is an equivalence relation, $E(X)$ contains only one minimal set I. Now I consists of one element. Thus (I, T) is a minimal transformation group in $(E(X), T)$. Therefore $(E(X), T)$ is proximal.)

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