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ON THE INFINITESIMAL VARIATIONS UNDER SOME CONDITIONS IN THE RIEMANNIAN MANIFOLD AND ITS SUBMANIFOLDS

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§0. Introduction

Recently infinitesimal variations of submanifolds of a Riemannian manifold have been studied by B.-Y. Chen [1], U-H. Ki [3], [4], [8], J.S. Pak [8], K. Yano [1], [5], [7], [8], the present author [4] and many others.

Moreover Yano, Ki and Pak in their paper [8] studied infinitesimal variations which preserves the Ricci tensor of a submanifold, and proved the following:

THEOREM A ([8]). *Let M^n be a simply connected and complete minimal submanifold of a space M^m of constant curvature. If M^n admits m-n linearly independent infinitesimal normal parallel variations preserving the Ricci tensor of M^n, then M^n is totally geodesic if $c \leq 0$, M^n is $S^n(r)$ or $S^p(r_1) \times S^{n-p}(r_2)$ if $c > 0$, where $S^n(r)$ denotes an n-sphere of radius $r > 0$.*

In the present paper we investigate some kinds of infinitesimal variations of compact submanifolds of a Riemannian manifold, and prove the Theorem A without minimal condition. The main result appears in Theorem 4.1.

In the preliminary section §1, we state the structure equations for submanifolds of a Riemannian manifold needed for the later discussion.

In §2, we consider the infinitesimal variations of submanifolds of a Riemannian manifold, recall the variations of the Christoffel symbols and those of the curvature tensor. When the submanifold is compact, or complete and irreducible, we obtain some global properties on various variations.

In §3, we study an infinitesimal variation which carries an Einstein space as a compact submanifold into an Einstein space.

In the last §4, we prove Theorem 4.1 and its corollary.

§1. Preliminaries ([3], [6])

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Let M^m be an m-dimensional Riemannian manifold covered by a system of the coordinate neighborhoods $\{U; x^h\}$ and denote by g_{ji}, $\left\{ {h \atop ji} \right\}$, ∇_j, $K_{kji}{}^h$ and k_{ji} the metric tensor, the Christoffel symbols formed with g_{ji}, the operator of covariant differentiation with respect to $\left\{ {h \atop ji} \right\}$, the curvature tensor and Ricci tensor of M^m, respectively, where the indices $h, i, j, k, \ldots$ run over the range $\{1, 2, 3, \cdots, m\}$ here and in the sequel.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and denote by g_{cb}, $\left\{ {a \atop cb} \right\}$, ∇_c, $K_{dcb}{}^a$ and K_{cb} the corresponding quantities of M^n respectively, where here and in the sequel the indices a, b, c, d, e run over the range $\{1, 2, 3, \cdots, n\}$.

Now we suppose that M^n is isometrically immersed in M^m by the immersion $i: M^n \to M^m$ and identify $i(M^n)$ with M^n itself, and we represent i by $x^h = x^h(y^a)$ and put $B_b{}^h = \partial_b x^h$, $(\partial_b = \partial/\partial y^b)$. Then $B_b{}^h$ are n linearly independent vectors of M^m tangent to M^n. Since the immersion i is isometric, we get

$$g_{ji} B_c^j B_b^i = g_{cb}. \tag{1.1}$$

We denote by $C_y{}^h$ $m-n$ mutually orthogonal unit normals to M^n, where here and in the sequel x, y, z run over the range $\{n+1, n+2, \cdots, m\}$. Then the metric tensor of the normal bundle of M^n is given by $\delta_{zy} = C_z^j C_y^i g_{ji}$.

It is well known that the Christoffel symbols $\left\{ {a \atop cb} \right\}$ and $\left\{ {h \atop ji} \right\}$ are related by

$$\left\{ {a \atop cb} \right\} = \left(\partial_c B_b{}^h + \left\{ {h \atop ji} \right\} B_c{}^j B_b{}^i\right) B^a{}_h, \tag{1.2}$$

where $B^a{}_h = B_b{}^i g^{ba} g_{ih}$, $(g^{ba}) = (g_{ba})^{-1}$.

Denoting by $\nabla_c B_b{}^h$ and $\nabla_c C_y{}^h$ the van der Waerden-Bortolotti covariant derivatives of $B_b{}^h$ and $C_y{}^h$ along M^n respectively, then we can write down the equations of Gauss and those of Weingarten in the form

$$\nabla_c B_b{}^h = h_{cb}{}^x C_x{}^h, \tag{1.3}$$

$$\nabla_c C_y{}^h = -h_c{}^a{}_y B_a{}^h \tag{1.4}$$

respectively, where $h_{cb}{}^x$ are the second fundamental tensors of M^n with respect to the normals $C_x{}^h$ and $h_c{}^a{}_x = h_{cbx} g^{ba}$.

The mean curvature vector of M^n is given by

$$H^h = \frac{1}{n} h_e{}^{ex} C_x{}^h. \tag{1.5}$$

From (1.4) and (1.5) we find

$$\nabla_c H^h = \frac{1}{n}(\nabla_c h^x)\, C_x{}^h - \frac{1}{n} h^x h_c{}^a{}_x B_a{}^h, \tag{1.6}$$

where $h^x = h_e{}^{ex}$. Thus, if the mean curvature of M^n is parallel in the normal bundle, we have

$$\nabla_c h^x = 0. \tag{1.7}$$

If the ambient manifold M^m is a space of constant curvature k, then the equations of Gauss, Codazzi and Ricci are respectively

$$K_{dcb}{}^a = k(\delta_d^a g_{cb} - \delta_c^a g_{db}) + h_d{}^a{}_x\, h_{cb}{}^x - h_c{}^a{}_x h_{db}{}^x, \tag{1.8}$$

$$\nabla_d h_{cb}{}^x - \nabla_c h_{db}{}^x = 0, \tag{1.9}$$

$$K_{dcy}{}^x = h_{de}{}^x h_c{}^e{}_y - h_{ce}{}^x h_d{}^e{}_y. \tag{1.10}$$

In the last equation, $K_{dcy}{}^x$ means the curvature tensor of the connection induced in the normal bundle.

§2. Variations of submanifolds.

We consider an infinitesimal variation of M^n in a Riemannian manifold M^m given by $\bar{x}^h = x^h(y) + \xi^h(y)\varepsilon$, where $\xi^h(y)$ is a vector field of M^m defined along M^n and ε is an infinitesimal. We then have

$$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n linearly independent vectors tangent to the varied submanifold. If we displace $\bar{B}_b{}^h$ parallely from the point $(\bar{x}^h)$ to (x^h), then we obtain the vectors

$$\begin{aligned} \tilde{B}_b{}^h &= \bar{B}_b{}^h + \left\{ {h \atop ji} \right\} (x + \xi\varepsilon)\xi^j B_b{}^i \varepsilon \\ &= B_b{}^h + (\nabla_b \xi^h)\varepsilon \end{aligned}$$

at the point (x^h), where

$$\nabla_b \xi^h = \partial_b \xi^h + \left\{ {h \atop ji} \right\} B_b{}^j \xi^i.$$

In the sequel we always neglect the terms of order higher than one with respect to ε. Thus by putting $\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h$, we have

$$\delta B_b{}^h = (\nabla_b\, \xi^h)\varepsilon. \tag{2.1}$$

If we put

$$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h, \tag{2.2}$$

then we get

(2.3) $$\nabla_b\xi^h=(\nabla_b\xi^a-h_b{}^a{}_x\xi^x)B_a{}^h+(\nabla_b\xi^x+h_{ba}{}^x\xi^a)C_x{}^h$$

because of (1.3) and (1.4).

When $\xi^a=0$, that is, when the variation vector ξ^h is normal to the submanifold M^n we say that the variation is *normal*, and when

(2.4) $$\nabla_b\xi^x+h_{ba}{}^x\xi^a=0,$$

that is, when the tangent space at a point (x^h) of the submanifold and that of the corresponding point $(\bar{x}^h)$ of the varied submanifold are parallel, we say that the variation is *parallel* [7]. Thus we see that the variation is parallel and normal if and only if

(2.5) $$\nabla_b\xi^x=0.$$

Now applying the operator δ to (1.1) and using (2.1), (2.3) and the fact that $\delta g_{ji}=0$, we get

(2.6) $$\delta g_{cb}=(\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x)\varepsilon,$$

where $\xi_b=g_{ba}\xi^a$, and further

(2.7) $$\delta g^{ba}=-(\nabla^b\xi^a+\nabla^a\xi^b-h^{ba}{}_x\xi^x)\varepsilon,$$

where $\nabla^a=g^{ca}\nabla_c$.

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be isometric and that for which δg_{cb} is proportional (with constant proportional factor) to g_{cb} is said to be *conformal* (or *homothetic*) [7]. In the sequel we put

(2.8) $$T_{cb}=\nabla_c\xi_b+\nabla_b\xi_c-2h_{cbx}\xi^x.$$

Then, we see from (2.6) that the following assertion holds: In order that a variation of a submanifold is isometric or conformal, it is necessary and sufficient that

(2.9) $$T_{cb}=0$$

or

(2.10) $$T_{cb}=2\lambda g_{cb}$$

for a certain function λ respectively.

The variation of the Christoffel symbols of M^n is given by [7]

(2.11) $$\delta\left\{{a \atop c\,b}\right\}=(\nabla_c\nabla_b\xi^a+K_{dcb}{}^a\xi^d)\varepsilon+\{\nabla_c(h_b{}^a{}_x\xi^x)+\nabla_b(h_c{}^a{}_x\xi^x)-\nabla^a(h_{cbx}\xi^x)\}\varepsilon,$$

which is equivalent to

$$(2.12)\qquad \delta\left\{{c \atop ab}\right\} = \{\nabla_c(\nabla_b\xi^a - h_b{}^a{}_x\xi^x) + \nabla_b(\nabla^a\xi_c - h_c{}^a{}_x\xi^x)$$
$$-\nabla^a(\nabla_b\xi_c - h_{bcx}\xi^x)\}\,\varepsilon$$

with the help of the Ricci identity.

A variation of a submanifold for which $\delta\left\{{a \atop cb}\right\}=0$ is said to be *affine* and that for which

$$(2.13)\qquad \delta\left\{{a \atop cb}\right\} = (\nabla_c\lambda)\delta_b^a + (\nabla_b\lambda)\delta_c^a - (\nabla^a\lambda)g_{cb}$$

is said to be *affine collinear* where λ being a certan function.

From (2.11) we hve

$$\nabla_c(\nabla_b\xi_a + \nabla_a\xi_b - 2h_{bax}\xi^x) = \left(\delta\left\{{e \atop cb}\right\}\right)g_{ea} + \left(\delta\left\{{e \atop ca}\right\}\right)g_{eb}.$$

By substituting (2.13) into this equation and taking account of (2.8), we find

$$(2.14)\qquad \nabla_c(T_{ba} - 2\lambda g_{ba}) = 0.$$

Thus if the submanifold is complete and irreducible, then we have

$$(2.15)\qquad T_{cb} = 2\mu g_{cb}$$

for some function μ.

Conversely, from (2.15) we can deduce $\nabla_c(T_{ba} - 2\mu g_{ba}) = 0$, which is equivalent to (cf. lemma 2.1 [3])

$$(2.16)\qquad \nabla_c\nabla_c\xi^a + K_{abcd}\xi^d$$
$$= \nabla_b(h_{cax}\xi^x + \mu g_{ca}) + \nabla_c(h_{bax}\xi^x + \mu g_{ba}) - \nabla_a(h_{cbx}\xi^x + \mu g_{cb})$$

with the help of the Ricci identity. Comparing (2.16) with (2.11), we see that the variation is affine collinear.

Thus, we have

PROPOSITION 2.1. *A variation of a complete and irreducible submanifold of a Riemannian manifold is affine collinear if and only if it is conformal.*

From (2.14) with $\lambda=0$ and (2.16) with $\mu=0$ we have immediately the following

PROPOSITION 2.2. *A variation of a submanifold of a Riemannian manifold satisfying $\nabla_c T_{ba}=0$ is affine.*

For a compact orientable submanifold M^n of a Riemannian manifold, we have the following integral formula:

$$\int [\{\nabla^e\nabla_e\xi^a + K_e{}^a\xi^e - 2\nabla^e(h_e{}^a{}_x\xi^x) + \nabla^a h_x\xi^x\}\xi_a + \frac{1}{2}\|T_{cb}\|^2 - \frac{1}{2}T_e{}^e(\nabla_b\xi^b) + T_{cb}h^{cb}{}_x\xi^x]\, dV = 0, \tag{2.17}$$

which is valid for any vector field ξ^a in M^n where dV being the volume element of M^n [7].

Now, if a variation is affine, we have $\nabla_c T_{ba}=0$ and (2.16) with $\mu=0$. Hence (2.17) becomes

$$\int \{\frac{1}{2}\|T_{cb}\|^2 + T_{cb}h^{cb}{}_x\xi^x\}\, dV = 0$$

because of $\nabla^b(T_e{}^e\xi_b) = T_e{}^e(\nabla_b\xi^b)$. Consequently, if $T_{cb}h^{cb}{}_x\xi^x \geq 0$ then we have $T_{cb}=0$ and hence the variation is isometric.

Therefore, we obtain

PROPOSITION 2.3. *If a variation of a compact orientable submanifold of a Riemannian manifold is affine and satisfies $T_{cb}h^{cb}{}_x\xi^x \geq 0$, then it is isometric.*

The variation of the curvature tensor $K_{dcb}{}^a$ of M^n is given by [8]:

$$\delta K_{dcb}{}^a = \nabla_d\left(\delta\left\{ {a \atop cb} \right\}\right) - \nabla_c\left(\delta\left\{ {a \atop db} \right\}\right).$$

Substituting (2.12) into this equation, we have

$$\begin{aligned}\delta K_{dcb}{}^a = [&\nabla_d\nabla_c(\nabla_b\xi^a - h_b{}^a{}_x\xi^x) - \nabla_c\nabla_d(\nabla_b\xi^a - h_b{}^a{}_x\xi^x)\\ &+ \nabla_d\{\nabla_c(\nabla^a\xi_c - h_c{}^a{}_x\xi^x) - \nabla^a(\nabla_b\xi_c - h_{bcx}\xi^x)\}\\ &- \nabla_c\{\nabla_b(\nabla^a\xi_d - h_d{}^a{}_x\xi^x) - \nabla^a(\nabla_b\xi_d - h_{dbx}\xi^x)\}]\varepsilon,\end{aligned} \tag{2.18}$$

from which we obtain

$$\begin{aligned}\delta K_{cb} = [&\nabla^e\nabla_c(\nabla_b\xi_e - h_{bex}\xi^x) + \nabla^e\nabla_b(\nabla_e\xi_c - h_{ecx}\xi^x)\\ &- \nabla^e\nabla_e(\nabla_b\xi_c - h_{bcx}\xi^x) - \nabla_c\nabla_b(\nabla_e\xi^e - h_e{}^e{}_x\xi^x)]\varepsilon.\end{aligned} \tag{2.19}$$

Transvection with g^{cb} yields

$$g^{cb}\delta K_{cb} = \{\nabla^c\nabla^b T_{cb} - \nabla^e\nabla_e T_b{}^b\}\varepsilon. \tag{2.20}$$

Thus, by (2.7) and (2.20), the variation of the scalar curvature $K=g^{cb}K_{cb}$ of M^n is given by the following form:

$$\delta K = (\delta g^{cb})K_{cb} + g^{cb}\delta K_{cb} = \{-T^{cb}K_{cb} + \nabla^c\nabla^b T_{cb} - \nabla^e\nabla_e T_b{}^b\}\varepsilon. \tag{2.21}$$

§3. Variations preserve Einstein space

An infinitesimal variation given by $\bar{x}^h = x^h + \xi^h\varepsilon$ is called *Einstein preser-*

ving if it carries an Einstein space into an Einstein space.

We now suppose that a variation is Einstein preserving. Then we have

$$\delta K_{cb}=\frac{1}{n}(\delta K)g_{cb}+\frac{K}{n}\delta g_{cb}.$$

Substituting (2.6), (2.19) and (2.21) into this equation and taking account of (2.8) and $K_{cb}=\frac{K}{n}g_{cb}$, we obtain

$$\begin{aligned}&\nabla^e\nabla_e(\nabla_b\xi_e-h_{bex}\xi^x)+\nabla^e\nabla_b(\nabla_e\xi_c-h_{ecx}\xi^x)\\&-\nabla^e\nabla_e(\nabla_b\xi_c-h_{bcx}\xi^x)-\nabla_c\nabla_b(\nabla_e\xi^e-h_e{}^e{}_x\xi^x)\\&=\frac{1}{n}\{-K/n\,T_e{}^e+\nabla^e\nabla^d T_{ed}-\nabla^e\nabla_e T_d{}^d\}g_{cb}+\frac{1}{n}KT_{cb},\end{aligned}$$

from which, taking the skew symmetric part, we obtain

$$\begin{aligned}(3.1)\qquad&\nabla^e\nabla_c T_{be}+\nabla^e\nabla_b T_{ce}-\nabla^e\nabla_e T_{cb}-\nabla_c\nabla_b T_e{}^e\\&=\frac{2}{n}\{-K/n\,T_e{}^e+\nabla^e\nabla^d T_{ed}-\nabla^e\nabla_e T_d{}^d\}g_{cb}+\frac{2}{n}KT_{cb},\end{aligned}$$

Transvecting with T^{cb}, we have

$$\begin{aligned}&2(\nabla^e\nabla_c T_{be})T^{cb}-(\nabla^e\nabla_e T_{cb})T^{cb}-(\nabla_c\nabla_b T_e{}^e)T^{cb}\\&=\frac{2}{n}(\nabla^e\nabla^d T_{ed})T_b{}^b-\frac{2}{n}(\nabla^e\nabla_e T_d{}^d)T_b{}^b+\frac{2}{n}K\|T_{cb}-\frac{1}{n}T_e{}^e g_{cb}\|^2.\end{aligned}$$

Thus we have

$$\begin{aligned}(3.2)\qquad\nabla^e W_e=&2(\nabla_c T_{be})(\nabla^e T^{cb})-\|\nabla_d T_{cb}\|^2-\frac{n+2}{n}(\nabla_b T_e{}^e)(\nabla_c T^{bc})\\&+\frac{2}{n}\|\nabla_d T_e{}^e\|^2+\frac{2}{n}K\|T_{cb}-\frac{1}{n}T_e{}^e g_{cb}\|^2,\end{aligned}$$

where we have put

$$\begin{aligned}W_e=&2(\nabla_c T_{be})T^{cb}-(\nabla_e T_{cb})T^{cb}-(\nabla^b T_d{}^d)T_{be}\\&-\frac{2}{n}(A^b T_{eb})T_d{}^d+\frac{2}{n}(\nabla_e T_d{}^d)T_b{}^b.\end{aligned}$$

Since $g(X,Y)=\frac{1}{4}\{\|X+Y\|^2-\|X-Y\|^2\}$ for any vector field X and Y on M^n, (3.2) reduces to

$$(3.3)\qquad \nabla^e W_e=\frac{1}{2}\|\nabla_d T_{cb}+\nabla_c T_{db}\|^2+\frac{n+2}{4n}\|\nabla_d T_e^e-\nabla_e T_d^e\|^2$$
$$+\frac{2}{n}\|\nabla_d T_e^e\|^2+\frac{2}{n}K\|T_{cb}-\frac{1}{n}T_e^e g_{cb}\|^2$$
$$-\{\|\nabla_d T_{cb}\|^2+\frac{1}{2}\|\nabla_d T_{cb}-\nabla_c T_{db}\|^2$$
$$+\frac{n+2}{4n}\|\nabla_d T_e^e+\nabla_e T_d^e\|^2\}.$$

Therefore, assuming

$$(3.4)\qquad 2n\|\nabla_d T_{cb}+\nabla_c T_{db}\|^2+(n+2)\|\nabla_d T_e^e-\nabla_e T_d^d\|^2$$
$$+8\|\nabla_d T_e^e\|^2+8K\|T_{cb}-\frac{1}{n}T_e^e g_{cb}\|^2\leq 0$$

and the submanifold M^n to be compact and applying Green's theorem, we obtain

$$(3.5)\qquad \nabla_d T_{cb}=0,\quad K\|T_{cb}-\frac{1}{n}T_e^e g_{cb}\|=0.$$

According to Proposition 2.2 and (3.5), we obtain the following

THEOREM 3.1. *In a compact submanifold of a Riemannian manifold, an infinitesimal Einstein preserving variation satisfying* (3.4) *is affine. Moreover, if the scalar curvature of the submanifold is negative, then the variation is homothetic.*

On the other hand, the variation of the volume element of M^n could be given by [7]

$$(3.6)\qquad \delta dV=(\nabla_e\xi^e-h_e{}^e{}_x\xi^x)\,dV\varepsilon=\frac{1}{2}T_e^e dV\varepsilon.$$

Taking account of (3.4) (3.5) and (3.6), we obtain

COROLLARY 3.2. *If an infinitesimal Einstein preserving variation of a compact submanifold of a Riemannian manifold is volume preserving and satisfies*

$$\|\nabla_d T_{cb}+\nabla_c T_{db}\|^2+\frac{4}{n}K\|T_{cb}\|^2\leq 0,$$

then it is affine. Moreover, if the scalar curvature of the submanifold with $K<0$, *then it is isometric.*

§4. Variations preserving the Ricci tensor.

In this section we suppose that the submanifold M^n admits $m-n$ linearly

from which, taking the skew-symmetric part with respect to d and c, and using (1.9), we have

$$(4.9)\qquad h^y h_c{}^e{}_y(\nabla_d h_{bex}) - h^y h_d{}^{ey}(\nabla_c h_{bex}) - (\nabla_d A_{yx}) h_{cb}{}^y + (\nabla_c A_{yx}) h_{db}{}^y = 0.$$

Interchanging indices b and d in (4.9), we get

$$(4.10)\qquad h^y h_c{}^e{}_y(\nabla_b h_{dex}) - h^y h_b{}^e{}_y(\nabla_c h_{dex}) - (\nabla_d A_{yx}) h_{cd}{}^y + (\nabla_c A_{yx}) h_{bd}{}^y = 0.$$

From (4.8) and (4.10), we have

$$h^x[2h^y h_c{}^e{}_y(\nabla_d h_{bex}) + nk\nabla_d h_{cbx} - A_{yx}(\nabla_d h_{cb}{}^y) - (\nabla_d A_{yx}) h_{cb}{}^y - (\nabla_b A_{yx}) h_{cd}{}^y + (\nabla_c A_{yx}) h_{bd}{}^y = 0$$

because of (1.9), and hence we obtain

$$(4.11)\qquad 2h^y h_c{}^e{}_y(\nabla_d h_{bex}) + nk(\nabla_d h_{cbx}) - A_{yx}(\nabla_d h_{cb}{}^y) - (\nabla_d A_{yx}) h_{cb}{}^y - (\nabla_b A_{yx}) h_{cd}{}^y + (\nabla_c A_{yx}) h_{bd}{}^y = 0$$

because of M^n admits $m-n$ linearly independent h^x.

Transvecting (4.11) with h^{cbx} and substituting (4.6), we obtain

$$2(-nkh^{bex} + A_y{}^x h^{bey} + kh^x g^{be})\nabla_d h_{bex} + nkh^{cbx}\nabla_d h_{cbx} - A_{yx}h^{cbx}\nabla_d h_{cb}{}^y - (\nabla_d A_{yx}) h_{cb}{}^y h^{cbx} = 0,$$

and further, using (1.7) and (4.7), we obtain

$$(4.12)\qquad -\frac{1}{2}nk\nabla_d(h_{cbx}h^{cbx}) + A_{yx}h^{cbx}\nabla_d h_{cb}{}^y - \frac{1}{2}\nabla_d(A_{yx}A^{yx}) = 0,$$

Moreover, from (4.7) we have

$$\begin{aligned}\nabla_d(A_{yx}A^{yx}) &= 2A_{yx}\nabla_d(h_{cb}{}^y h^{cbx}) \\ &= 2A_{yx}\{(\nabla_d h_{cb}{}^y) h^{cbx} + h_{cb}{}^y(\nabla_d h^{cbx})\} \\ &= 4A_{yx}h^{cbx}(\nabla_d h_{cb}{}^y).\end{aligned}$$

Thus (4.12) reduces to

$$2nk\nabla_d(h_{cbx}h^{cbx}) + \nabla_d(A_{yx}A^{yx}) = 0,$$

which implies that

$$2nkh_{cbx}h^{cbx} + A_{yx}A^{yx} = \text{constant}.$$

Covariant differentiation of this equation yields

$$2nk(\nabla_a h_{cbx})h^{cbx}+(\nabla_a A_{yx})A^{yx}=0, \tag{4.13}$$

from which, taking account of (4.3), we have

$$2nk\|\nabla_a h_{cbx}\|^2+\|\nabla_a A_{yx}\|^2+(\nabla^a\nabla_a A_{yx})A^{yx}=0. \tag{4.14}$$

On the other hand, by using (4.3) we obtain

$$\begin{aligned} A^{yx}(\nabla^a\nabla_a A_{yx}) &= A^{yx}\nabla^a\{(\nabla_a h_{cby})h^{cb}{}_x+h_{cby}(\nabla_a h^{cb}{}_x)\} \\ &= 2(\nabla_a h_{cby})(\nabla^a h^{cb}{}_x)A^{yx} \\ &= 2(\nabla_a h_{cby})(\nabla^a h^{cb}{}_x)h^{dex}h_{de}{}^y=2\|(\nabla_a h_{cby})h_{de}{}^y\|^2. \end{aligned}$$

Thus (4.14) becomes

$$2nk\|\nabla_a h_{cbx}\|^2+\|\nabla_a A_{yx}\|^2+2\|(\nabla_a h_{cby})h_{de}{}^y\|^2=0. \tag{4.15}$$

Consequently we see that

$$(\nabla_a h_{cby})h_{de}{}^y=0$$

since $k\geq 0$. It follows that

$$\nabla_a h_{cb}{}^x=0$$

because of (4.3). Thus, the Ricci tensor of M^n is covariantly constant.

This fact is very interesting in various meanings [2], and now we have following theorem:

THEOREM 4.1. *Let M^n be a simply connected and complete submanifold of a space M^m of a constant curvature $k\geq 0$ such that the mean curvature vector $H^h=\frac{1}{n}h^x C_x{}^h$ is parallel in the normal bundle and M^n admits $m-n$ linearly independent h^x. If M^n admits $m-n$ linearly independent infinitesimal normal parallel variations preserving the Ricci tensor of M^n, then M^n is $S^n(r)$ or $S^p(r_1)\times S^{n-p}(r_2)$ if $k>0$, M^n is $S^{n-p}\times R^p$ if $k=0$, where $S^n(r)$ denotes an n-sphere of radius $r>0$ and R^p an p-dimensional space of curvature 0.*

COROLLARY 4.2. *Let M^n be a complete submanifold of a unit sphere $S^m(1)$ such that the mean curvature vector $H^h=1/nh^x C_x{}^h$ is parallel in the normal bundle and M^n admits $m-n$ linearly independent h^x. If M^n admits $m-n$ linearly independent infinitesimal normal parallel variations preserving the Ricci tensor of M^n, then M^n is a sphere S^n or $S^p\times S^{n-p}$.*

In addition, from (4.3) we see

$$\nabla^e(\nabla_e h_{cbx}h^{cbx})=\|\nabla_e h_{cb}{}^x\|^2,$$

and obtain the following

PROPOSITION 4.3. *Let M^n be a compact submanifold of a unit sphere $S^m(1)$ such that the mean curvature vector is parallel in the normal bundle. If M^n admits m−n linearly independent infinitesimal normal parallel variations preserving the Ricci tensor of M^n, then M^n is a sphere S^n or $S^p \times S^{n-p}$.*

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