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NOTE ON THE PRIME RADICAL IN NONASSOCIATIVE RINGS

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1. The prime radical

Several definitions of prime ideals in a nonassociative ring have been introduced during the past decade. An axiomatic definition, based on a $*$-operation, is given in [3] and extends most of the known results for the prime radical [4]. A $*$-operation in a nonassociative ring R is a mapping of $I(R)\times I(R)$, where $I(R)$ is the lattice of ideals in R, into the lattice of additive subgroups of R such that, for $A,B,C,D\in I(R)$, 1) if $A\subseteq C$ and $B\subseteq D$, then $A*B\subseteq C*D$, 2) $(0)*A=B*(0)=(0)$, and 3) if $\bar{R}$ is a homomorphic image of R, then $\overline{A*B}=\bar{A}*\bar{B}$. An ideal P of R is called $*$-prime if $A*B\subseteq P$ for $A,B\in I(R)$ implies that $A\subseteq P$ or $B\subseteq P$. A nonempty subset M of R is called a $*$-system if, for A, $B\in I(R)$, $M\cap A\neq\phi$ and $M\cap B\neq\phi$ imply $A*B\cap M\neq\phi$. The $*$-prime radical $P^*(R)$ of R is defined to be the set of elements $x\in R$ such that any $*$-system containing x also contains 0 and shown to be the intersection of all $*$-prime ideals in R. If R is an s-ring for a positive integer $s\geq 2$, then there exists a $*$-operation in R such that $A*A=A^s$ for all $A\in I(R)$ and $P^*(R)$ coincides with the prime radical $P(R)$ of R defined by Zwier [8] ([4]).

Let $A*B=AB^2+(AB)B+B(AB)+(BA)B+B(BA)+B^2A$ for A, $B\in I(R)$. Then, in a weakly W-admissible ring R, $A*B$ is also an ideal of R [7]. The proof of Smith [6, Lemma 2.3] can be applied to show that $A*B$ for A, $B\in I(R)$ is an ideal in a generalized alternative ring II. Thus these rings are 3-rings which generalize Lie, alternative, Jordan, standard, and generalized standard rings. If we let $A\circ B=AB+BA$ for A, $B\in I(R)$, it is shown that $A\circ B$ is an ideal in Lie, alternative and $(-1,1)$ rings [1] (in the alternative case, AB is an ideal). Hence these rings are 2-rings. In a 2-ring we have following

PROPOSITION 1. *Let* $A*B=AB^2+B^2A+(AB)B+B(AB)+(BA)B+B(BA)$ *and* $A\circ B=AB+BA$ *for* A, $B\in I(R)$. *In a 2-ring* R, *an ideal* P *of* R *is prime if and only if* P *is* $*$-*prime if and only if* P *is* $\circ$-*prime.*

Proof. Let P be prime and let $A*B\subseteq P$ for $A,B\in I(R)$. Then $AB^2\subseteq A*B\subseteq P$. Since B^2 is an ideal of R and P is prime, $A\subseteq P$ or $B\subseteq P$ and so P

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is $*$-prime. Suppose that P is $*$-prime and $AB \subseteq P$. Let $C = A \cap B$. Then $C*C = C^2C + CC^2 \subseteq A^2B + AB^2 \subseteq AB \subseteq P$ and so $C \subseteq P$. Thus $A*B \subseteq A \cap B = C \subseteq P$, and since P is $*$-prime, $A \subseteq P$ or $B \subseteq P$ and P is prime. If P is prime, it is clearly $\circ$-prime. Suppose that P is $\circ$-prime and $AB \subseteq P$. By a similar argument, if we let $C = A \cap B$, then $C \circ C = C^2 \subseteq AB \subseteq P$ and so $C \subseteq P$. It follows from this that $BA \subseteq A \cap B = C \subseteq P$ and $A \circ B \subseteq P$. Thus P is prime.

Proposition 1 has been proved in a ring R in which AB is an ideal for A, $B \in I(R)$ [4, Lemma 3.2].

In this note we give an analogous characterization of the prime radical in an s-ring for the $*$-prime radical of any rings [5]. We make use of this to show that, in an s-ring R, every nonzero ideal of R which is contained in the prime radical of R contains a nonzero ideal K of R such that $K^s = 0$, and that the prime radical of R is essentially nilpotent. This extends the result of Fisher [2] for associative rings to any s-ring.

2. Characterization of the prime radical

Following Rich [5], we make

DEFINITION 1. Let R be any ring equipped with a $*$-operation. A sequence $\{a_0, a_1, \cdots, a_n \cdots\}$ in R is called a P^*-sequence if $a_n \in (a_{n-1})*(a_{n-1})$ for $n = 1, 2, \cdots$. An element a of R is called strongly $*$-nilpotent if every P^*-sequence beginning with a is ultimately 0.

If R is an s-ring in which $A*A = A^s$ for $A \in I(R)$, then the P^*-sequences are the P-sequences in [5].

THEOREM 2. *The $*$-prime radical $P^*(R)$ of any ring R is the set of all strongly $*$-nilpotent elements in R.*

Proof. Let a be an element in R but not in $P^*(R)$. Then there exists a $*$-prime ideal P of R which does not contain a. The complement $c(P)$ of P is a $*$-system in R. Let $a_0 = a$. Since $(a_0) \cap c(P) \neq \phi$, there exists a nonzero element a_1 in $(a_0)*(a_0) \cap c(P)$, and we inductively find a sequence $S = \{a_0, a_1, \cdots, a_n, \cdots\}$ in R such that $a_{n+1} \in (a_n)*(a_n) \cap c(P)$. Thus S is a P^*-sequence beginning with a which does not end in zero, so that a is not strongly $*$-nilpotent in R.

Conversely, suppose that $a \in P^*(R)$ and that $S = \{a_0, a_1, \cdots, a_n, \cdots\}$, where $a_0 = a$, is a P^*-sequence beginning with a. Let A, B be ideals of R such that $A \cap S \neq \phi$ and $B \cap S \neq \phi$. There exist elements $a_{i_1} \in A \cap S$, $a_{i_2} \in B \cap S$. Let $j = \max\{i_1, i_2\}$. Then $a_{j+1} \in (a_j)*(a_j) \subseteq (a_{i_1})*(a_{i_2}) \subseteq A*B$. Thus $a_j \in A*B \cap S \neq \phi$, and this shows that S is a $*$-system in R. Since $a \in S \cap P^*(R)$, S must contain 0. Hence $a_j=0$ for some j and a is strongly $*$-nilpotent.

COROLLARY 3. *Let J be a nonzero ideal of R which is contained in $P^*(R)$. Forevery $*$-operation in R, J contains a nonzero ideal K of R such that $K*K=0$.*

Proof. Let a be a nonzero element in J. If $(a)*(a)\neq 0$, there exists a nonzero element $a_1\in(a)*(a)\subseteq(a)\subseteq J$. If $(a_1)*(a_1)\neq 0$, then by Theorem 2 we can continue this to obtain a nonzero element $a_{n+1}\in(a_n)*(a_n)\subseteq J$ such that $(a_{n+1})*(a_{n+1})=0$.

If R is an s-ring, there exists a $*$-operation in R such that $A*A=A^s$ for every $A\in I(R)$. Hence we have

COROLLARY 4. *Each nonzero ideal J of an s-ring R which is contained in the prime radical $P(R)$ contains a nonzero nilpotent ideal K of R such that $K^s=0$.*

If R is a Lie, alternative or $(-1, 1)$ ring (a 2-ring) then each nonzero ideal of R which is contained in the prime radical contains a nonzero ideal K of R such that $K^2=0$. This improves the result of Fisher [2] for associative rings, which requires the additional assumption that the ring has an identity.

DEFINITION 2. An ideal K of R is said to be essentially nilpotent if K contains a nilpotent ideal L of R which is essential in K, i.e., L has nonzero intersection with nonzero ideal of R contained in K.

Note that every nonzero nilpotent ideal of R is essentially nilpotent. While it is well-known that the prime radical $P(R)$ of an s-ring R contains all nilpotent ideals of R, it is not known whether $P(R)$ is nilpotent even under the chain condition on one-sided ideals. However we can show that $P(R)$ is essentially nilpotent. This has been proved for associative rings [2].

THEOREM 5. *Let R be an s-ring. Every nonzero ideal J of R which is contained in the prime radical $P(R)$ of R is essentially nipotent.*

Proof. The proof proceeds as in [2]. Let $\{N_t \mid t\in T\}$ be the collection of all nonzero nilpotent ideals N_t of R such that $N_t\subseteq J$ and $N_t^s=0$. By Corollary 4 this collection is not empty. Let $\Omega=\{S\subseteq T \mid \sum_{i\in S}N_i \text{ is direct}\}$. Then Ω is non-empty and inductive. Hence by Zorn's lemma one finds a maximal element U in Ω. Let $N=\sum_{i\in U}N_i$. Since the sum is direct and each N_i is an ideal, we have that $N^s=0$. We show that N is essential in J. If not, then there exists a nonzero ideal $K\subseteq J$ of R such that $N\cap K=0$. Corollary 4 then ensures that there exists a nonzero $N_t\subseteq K$ for some $t\in T$ such that

$N_t'=0$. Hence $N+N_t$ is direct and this contradicts the maximality of U. Therefore, N is essential in J and J is essentially nilpotent.

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