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FINITE MODULES AND COMBINTORICS

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In other papers, e.g. [2], we have been interested in relating the structure of rings to the structure of its modules. In any case, this has proven to be a successful procedure in many circumstances as is clear from myriad examples in algebra. In this paper we study the following problems.

Problem A: Suppose that R is a ring with identity. Suppose furthermore that for any positive integer m, the collection of isomorphism classes of left unitary R-modules M such that M contains m elements is finite. Let λ_R denote the function with domain the positive integers such that $\lambda_R(m)$ is this number. Then describe general properties of the function λ_R and give methods by which λ_R may be determined.

Problem B: With the situation as in Problem A, determine to which extent the ring R is known from the function λ_R. In particular, if R and S are rings such that $\lambda_R=\lambda_S$, determine how R and S may be related.

Roughly speaking, if R is the type of ring mentioned in these problems, then the question is to investigate how much is known about R from the function λ_R and vice-versa. This is a problem of great complexity as is clear from the literature [*cf.* 1, for references]. Indeed, great progress in this area would be made if one could determine the class of finite indecomposable R-modules associated with a ring R. This is known to be a difficult problem. Nevertheless progress can be made, as we intend to show in this paper.

If R is a ring of the type discussed, then we shall call R *a left-combinatorial ring*. In Theorem 9 we identity left combinatorial rings. It turns out that notion is symmetric, i.e., if one defines *right-combinatorial rings* with respect to right unitary R-modules, then R is left-combinatorial if and only if R is right-combinatorial.

The function λ_R is the (*left*) *partition function of* R. We introduce a *radical* $K(R)$, the intersection of all ideals of finite index in R, and establish several of its properties. In theorem 10 we identity those rings R for which $K(R)=0$. We observe that if R is left-combinatorial and if $S=R/K(R)$, then $\lambda_R=\lambda_S$. Thus in Problem B we have a result involving $K(R)$. The function λ_R is multiplicative, i.e., if $(m,n)=1$, then $\lambda_R(mn)=\lambda_R(m)\lambda_R(n)$.

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Let $\lambda_R^*(m)$ denote the number of non-isomorphic indecomposable modules M such that $|M|=m$. Here we let $\lambda_R^*(1)=1$, i.e., we count 0 as an indecomposable module. This function is the *second partition function of R*. In Theorem1 we show that there is a fixed relation, i.e. independent of the ring R, between λ_R and λ_R^*. Thus knowledge about one yields knowledge about the other. In Theorem 2 we relate λ_R and λ_S, where $S=R_n$, the complete ring of matrices with coefficients in R. In Theorem 3 we relate λ_R and λ_A, λ_B, where $R=A\oplus B$ is the direct sum of rings. In Theorem 7 we identify those rings R such that $\lambda_R(1)=1$ and $\lambda_R(m)=0$ otherwise. In Theorem 8 we identify those rings R such that $\lambda_R(m)\leq 1$ for all m.

At least in these cases the combinatorial conditions are algebraically identifiable, contributing to information about both Problem A and Problem B.

We believe that further investigation of these problems would provide opportunities for an interesting blending of results from ring theory and the theory of partitions or more generally combinatorics.

Some properties of partition functions.

If $|M|=m$, and $m=ab$, where $(a,b)=1$ and $a\geq 2$, $b\geq 2$, then $M=aM\oplus bM$, and by the fundamental theorem of abelian groups $aM\neq 0$, $bM\neq 0$, so that M is not indecomposable, and $\lambda_R^*(m)=0$.

Also, since given M, with $|M|=m$, this decomposition is unique with $|bM|=a$, $|aM|=b$, it follows that $\lambda_R(m)=\lambda_R(a)\lambda_R(b)$, i.e., the function λ_R is a multiplicative function.

Since any finite R-module M is obviously Artinian and Noetherian, it follows by the Krull-Schmidt theorem that if $M=M_1\oplus\cdots\oplus M_k=N_1\oplus\cdots\oplus N_l$, with the M_i and N_j indecomposable R-modules, then $k=l$ and without loss of generality $M_i=N_i$ for all i, $1\leq i\leq k$.

Since modules $M\oplus N$ and $N\oplus M$ are isomorphic, the connection between the functions λ_R^* and λ_R can be described completely by the following scheme. Similar schemes are discussed in another paper by the author [3].

Let P be the set of positive integers and define products $P^k\times P^k\to P$ as follows:

$$(e_1,\cdots,e_k)\ \pi\ (n_1,\cdots,n_k)=n_1^{e_1}n_2^{e_2}\cdots n_k^{e_k}$$

$$(e_1,\cdots,e_k)\times(n_1,\cdots,n_k)=n_1^{(e_1)}\cdots n_k^{(e_k)},$$

where $$n^{(e)}=\binom{n}{1}\binom{e-1}{0}+\cdots+\binom{n}{e}\binom{e-1}{e-1}=\binom{n+e-1}{e}.$$

Thus $n^{(e)}$ is equal to the number of ordered partitions of $n-1$ into $e+1$ non-negative integers.

If $f: P \to P \cup \{0\}$ is any function whatsoever, we define

$$(e_1, \cdots, e_k) \times_f (n_1, \cdots, n_k) = (e_1, \cdots, e_k) \times (f(n_1), \cdots, f(n_k)),$$

where $0^{(e)} = 0$. Thus, e.g., if $\lambda_R^*: P \to P \cup \{0\}$ is defined as above, and $n_1 > n_2 > \cdots > n_k$, then

$$(e_1, \cdots, e_k) \times_{\lambda_R^*} (n_1, \cdots n_k) = \lambda_R^*(n_1)^{(e_1)} \cdots \lambda_R^*(n_k)^{(e_k)},$$

is the number of R-modules M such that M has $e_1 + \cdots + e_k$ indecomposable direct summands of which precisely e_j direct summands contain n_j elements, and where we use the convention that if $\lambda_R^*(n_i) = 0$,

$$(e_1, \cdots, e_k) \times_{\lambda_R^*} (n_1, \cdots, n_k) = 0.$$

That this is so follows from the argument given in [3, p. 164] or a sequence of straight forward observations.

We shall call a vector $\vec{n}$ in P^k *cascading* if $\vec{n} = (n_1, n_2, \cdots, n_k)$ with $n_1 > n_2 > \cdots > n_k$. A *logarithmic bipartition* of m is a pair $(\vec{e}, \vec{n}) \in P^k \times P^k$, for some k, such that $\vec{n}$ is cascading and such that $\vec{e} \amalg \vec{n} = m$.

Thus, if we consider the sum over all logarithmic bipartitions of m, $\sum_{\vec{e} \amalg \vec{n} = m} \vec{e} \times_{\lambda_R^*} \vec{n}$, then this number is $\lambda_R(m)$, the number of R-modules M such that $|M| = m$. In other words, our "functional equation" involving λ_R and λ_R^* is the following:

$$\begin{aligned} l_R(z) &= \textstyle\sum_{m=1}^{\infty} \lambda_R(m) z^m \\ &= \textstyle\sum_{m=1}^{\infty} (\sum_{\vec{e} \amalg \vec{n} = m} \vec{e} \times_{\lambda_R^*} \vec{n}) z^m. \end{aligned}$$

For arbitary functions $f: P \to P$, let us denote by $Df: P \to P$ the function $(Df)(m) = \sum_{\vec{e} \amalg \vec{n} = m} \vec{e} \times_f \vec{n}$. We want to invert this expression, i. e., we want to find polynomials $g_m(x_1, \cdots, x_m)$ for all m, such that $f(m) = g_m(Df(1), Df(2), \cdots, Df(m))$. Since $Df(1) = f(1)$, it suffices to take $g_1(x_1) = x_1$. Suppose now that $g_k(x_1, \cdots, x_k)$ has been defined for $k < m$.

Now, $Df(m) = \sum_{\vec{e} \amalg \vec{n} = m} \vec{e} \times_f \vec{n} = 1 \times_f m + \sum_{\vec{e} \amalg \vec{n} = m} \vec{e} \times_f \vec{n}$, where the sum $\sum^*$ runs over all logarithmic bipartitions $\vec{e} \amalg \vec{n} = m$ with $\vec{e} \neq (1)$, whence $\vec{n} \neq (m)$.

Hence $Df(m) = f(m) + \sum^*_{\vec{e} \amalg \vec{n} = m} \vec{e} \times_f \vec{n}$ and $f(m) = Df(m) - \sum_{\vec{e} \amalg \vec{n} = m} \vec{e} \times_f \vec{n}$. For polynomials $\phi(x_1, \cdots, x_m)$, define

$$\binom{\phi(x_1, \cdots, x_m)}{j} = \frac{1}{j!} \phi(x_1, \cdots, x_m) \cdot \cdots \cdot (\phi(x_1, \cdots, x_m) + 1 - j).$$

Thus also, $\phi(x_1, \cdots, x_m)^{(e)} = \binom{\phi(x_1, \cdots, x_m)+e-1}{e}$.

Now $f(n_i) = g_{n_i}(Df(1), \cdots, Df(n_i))$ has been defined for $n_i < m$, and thus:

$$\vec{e} \times_f \vec{n} = g_{n_1}(Df(1), \cdots, Df(n_1))^{(e_1)} \cdots g_{n_k}(Df(1), \cdots Df(n_k))^{(e_k)}.$$

If we let $\vec{e} \times \vec{n} = g_{n_1}(x_1, \cdots x_{n_1})^{(e_1)} \cdots g_{n_k}(x_1, \cdots, x_{n_k})^{(e_k)}$, then it follows readily that

$$g_m(x_1, \cdots, x_m) = x_m - \sum_{\vec{e}\cdot\vec{n}=m} \vec{e} \times \vec{n}$$

is ths desired polynomial.

We notice that $g_m(x_1, \cdots, x_m)$ is independent of the function f and that the polynomials $g_1(x_1)$, $g_2(x_1, x_2)$, $\cdots$ etcetera, can be computed directly.

Of course, if we define $\hat{D}f: P \to P$ by $\hat{D}f(m) = g_m(f(1), \cdots, f(m))$, then $\hat{D}Df = D\hat{D}f = f$, so that in particular, regarding λ_R and $\lambda_R{}^*$ as functions from P to P, we have:

$$l_R{}^*(z) = \sum_{m=1}^{\infty} \lambda_R{}^*(m) z^m = \sum_{m=1}^{\infty} g_m(\lambda_R(1), \cdots \lambda_R(m)) z^m.$$

We have proven the following theorem.

THEOREM 1: *Suppose that R is a ring and that $\lambda_R{}^*$ is a finite number for each positive integer m. Then $\lambda_R(m)$ is a finite number for each m and $\lambda_R = D\lambda_R{}^*$. Conversely, if $\lambda_R(m)$ is finite for each m, then $\lambda_R{}^*(m)$ is finite for each m and $\lambda_R{}^* = D\lambda_R$.*

Hence we refer of course to the notation just developed.

Notice also that $\lambda_R(1) = (1) \times (\lambda_R{}^*(1)) = \lambda_R{}^*(1)$, i.e., we count 0 as an indecomposable module.

Some examples

(a) If $R = \mathbf{Z}$, the ring of integers, then $\lambda_R{}^*(p_i) = 1$ and $\lambda_R{}^*(m) = 0$ if m is not the power of a prime. It is also true that $\lambda_R(p_1{}^{e_1} \cdots p_k{}^{e_k}) = \pi(e_1) \cdots \pi(e_k)$, where $\pi(a)$ is the number of partitions of the integer a. Thus, $\lambda_R{}^*$ is the characteristic function of the powers of the primes and we obtain a relation between this characteristic function and products of partition functions via the operators D and $\hat{D}$.

(b) If $R = \mathbf{Z}/(p^n)$, where $\mathbf{Z}$ is the ring of integers and p is a prime, then the indecomposable modules are 0, $\mathbf{Z}/(p), \mathbf{Z}/(p^2), \cdots, \mathbf{Z}/(p^n)$, so that $\lambda_R{}^*(1) = \lambda_R{}^*(p) = \lambda_R{}^*(p^2) = \cdots = \lambda_R{}^*(p^n) = 1$, and $\lambda_R{}^*(m) = 0$ otherwise.

(c) Let $T_2(S)$ be the ring of 2×2 matrices X with coefficients in S, such that $X_{12} = 0$, i.e., $T_2(S)$ is the collection of lower triangular matrices with coefficients in S. Let $R=T_2(\mathbf{Z}/(p^n))$ and let k be any positive integer. If $n \geq 4$, then there is an indecomposable R-module with a minimal generating set of $3k$ elements. In particular, R is a finite ring with an infinite number of indecomposables. Of course if R is finite, then $\lambda_R^*(m)$ and $\lambda_R(m)$ are finite for each integer m.

This example is discussed in MacDonald [1; p. 226] and is due to Brenner. Similarly, if $n \leq 3$, then $R=T_2(\mathbf{Z}/(p^n))$ has only a finite number of isomorphism classes of finitely generated indecomposable R-modules. In particular, $\sum_{m=1}^{\infty}\lambda_R^*(m)<\infty$ in this case. This example is discussed in [1; p. 243].

(d) Let G be the direct product of two cyclic groups of order p and let k be a finite field of characteristic p. Then if n is a positive integer, $k[G]$, the group algebra, has an indecomposable module of k-dimension $2n+1$. Thus in this case $\lambda_R^*(p^{2n+1}) \geq 1$. This example is also discussed in MacDonald [1; p. 228] and is due to Heller and Reiner.

Along the lines of 'general theory' we prove the following relatively simple result.

THEOREM 2: *Let R be a left combinatorial ring, and let R be the complete ring of $n \times n$ matrices over R. Then, there is a bijection between the class of R-modules M such that $|M|=m$ and the class of R_n-modules M^* such that $|M^*|=m^n$. This bijection maps indecomposable R-modules to indecomposable R_n-modules and conversely. In particular, R_n is also a left combinatorial ring. Furthermore, it follows that $\lambda_{R_n}^*(m^n)=\lambda_R^*(m)$ and $\lambda_{R_n}(m^n)=\lambda_R(m)$.*

Proof: Given an R-module M, let $M^n=\{(m_1, \cdots, m_n) \mid M_i \in M\}$. Then M^n can be regarded as an R_n-module by defining $(r_{ij})(m_1, \cdots, m_n)=(\omega_1, \cdots, \omega_n)$ where $\omega_i=\sum_{j=1}^{n} r_{ij}m_j$.

Conversely, given an R_n-module N, let $M=E_{11}N=E_{12}N=\cdots=E_{1n}N$, where E_{ij} is the matrix with 1 in position (i, j) and 0's elsewhere. Then it is easy to see that for each R-module M, $M \cong E_{11}M^n$ as R-modules. Moreover, for each R_n-module N, the mapping $f: N \to (E_{11}N)^n$ defined by $f(x)=(E_{11}x, E_{12}x, \cdots, E_{1n}x)$ is an R_n-isomorphism.

If $N=N_1 \oplus N_2$ and if M_i corresponds to N_i in the fashion indicated above, then $M=M_1 \oplus M_2$ corresponds to N. Similarly if $N_1 \cong N_2$, then $M_1 \cong M_2$ as well. It follows that if we let M^* be N, then the statements for the theorem have been demonstrated.

The following theorem is along the same lines and also proven without difficulty.

THEOREM 3: *Let $R=R_1 \oplus R_2$ be a direct sum of rings. Then R is a left combinatorial ring if and only if both R_1 and R_2 are left combinatorial rings.*

Furthermore, when this is the case,

$$\lambda_R(m)=\sum_{d\mid m}\lambda_{R_1}(d)\lambda_{R_2}(m/d).$$

Similarly,

$$\lambda_R^*(m)=\sum_{d\mid m}\lambda_{R_1}^*(d)\lambda_{R_2}^*(m/d).$$

Proof. Suppose that M is an R-module. We let $M_i=R_iM$. Then, $M=M_1\oplus M_2$ as a direct sum of abelian groups. Now $R_2M_1=0$ and $R_1M_2=0$, and thus $M=RM=R_1M_1\oplus R_2M_2=M_1\oplus M_2$, i. e., M_i is naturally an R_i-module. Hence, each R-module determines an ordered pair (M_1, M_2) of abelian groups, where M_i is an R_i-module. Conversely, let M_i be an arbitrary R_i-module, $i=1,2$ and define an R-module M by $M=M_1\oplus M_2$, with $(r_1+r_2)(m_1+m_2)=r_1m_1+r_2m_2$, with $r_i\in R_i$, $m_i\in M_i$. It follows readily that this mapping is a 'bijection' in that the constructions given are inverses of one another (up to isomorphism). Hence if M_1 is an R_1-module with d elements and M_2 is an R_2-module with m/d elements, then M is an R-module with $d(m/d)=m$ elements.

The first formula is now immediate. If $M=A\oplus B$, as R-modules, then $M_1=A_1\oplus B_1$, $M_2=A_2\oplus B_2$, and $M=(A_1\oplus B_2)\oplus(B_1\oplus B_2)$.

If M is indecomposable as an R-module, then it follows that both M_1 and M_2 are indecomposable as R_1 and R_2-modules respectively. Indeed, suppose $M_1=A_1\oplus B_1$, without loss of generality. Then $M=M_1\oplus M_2=(A_1\oplus M_2)\oplus(B_1\oplus 0)$ is a nontrivial direct sum decomposition of M as a direct sum of R-modules. The second formula follows.

Thus, e. g., if $R_1=R_2=S=\mathbf{Z}/(p^n)$, then if $R=R_1\oplus R_2$,

$$\lambda_R^*(p^m)=\sum_{i=0}^{t}\lambda_S^*(p^i)\lambda_S^*(p^{m-i}),$$

where $t=\min(n,m)$, and where $\lambda_S^*(p^j)=1$ if and only if $0\leq j\leq n$, by example (b). Observe the difference between this example and example (c) above.

Some properties preserved under epimorphisms.

Given a function f with domain containing a subset S of the positive integrs and range the positive integers, let $R_f(S)$ be the class of all rings R such that $\lambda_R(m)\leq f(m)$ for all positive integers $m\in S$.

Suppose that $\phi: R_1\rightarrow R_2$ is a (ring)epimorphism. If M is an R_2-module, then defining $rm=\phi(r)m$ for $r\in R_1$, it follows that M is an R_1-module. Notice that $\phi(1_{R_1})=1_{R_2}$ and $1_{R_1}m=1_{R_2}m=m$, so that we operate within our initial instructions.

Now, using this mechanism, it follows that if M_1 and M_2 are isomorphic as R_2-modules, then they are also isomorphic as R_1-modules. On the other hand, if they are not isomorphic as R_2-modules, they are not isomorphic as R_1-modules. Indecomposability is handled the same way. Accordingly it follows that if $\lambda_{R_1}(m)$ is bounded, then so is $\lambda_{R_2}(m)$ and $\lambda_{R_2}(m) \leq \lambda_{R_1}(m)$. Similarly, if $\lambda_{R_1}{}^*(m)$ is bounded, then so is $\lambda_{R_2}{}^*(m)$ and $\lambda_{R_2}{}^*(m) \leq \lambda_{R_1}{}^*(m)$.

THEOREM 4: *Suppose that $\phi: R_1 \to R_2$ is a (ring) epimorphism. Then if $R_f(S)$ is as defined above and if $R_1 \in R_f(S)$, $R_2 \in R_f(S)$ as well. Furthermore, if R_1 is left combinatorial, then R_2 is left combinatorial.*

Proof: The proofs of the statements in this theorem are all easy consequences of the observations made above, along with the definitions of various sets.

Abstractly, we may think of every class $R_f(S)$ as a class of rings having certain properties in common, i. e., those defining the class. This way we obtain a large class of properties 'closed' under taking epimorphic images. The class of left combinatorial rings is closed under direct sums, the forming of complete matrix rings and under taking epimorphic images. Thus it has much in common with a variety of rings, except that we do not require that the class be closed under formation of direct products (an 'infinite' operation), and that we do not require that the class be closed under formation of subrings, since we do not know whether this is actually the case for left combinatorial rings, or the circumstances under which this might follows.

A radical for rings.

Suppose that R is a ring. Suppose that F is the collection of all ideals of R of finite index, i. e., F consists of all ideals I of R such that R/I is finite. Then, we define $K(R)$, to be the intersection $K(R) = \cap \{I \mid I \in F\}$. Notice that if $K(R) = R$, then R contains no proper ideal of finite index and conversely. Thus, e. g., if $\mathbf{Q}$ is the field of rationals, then $K(\mathbf{Q}) = \mathbf{Q}$. On the other hand, if $\mathbf{Z}$ is the ring of integers, then $K(\mathbf{Z}) = (0)$. Indeed, if $m \neq 0$, and if $n > |m|$, then $m \notin (n)$.

THEOREM 5: *Given any ring R, $K(R/K(R)) = 0$.*

Proof: If $K(R) = R$, then $R/K(R) = 0$ and $K(R/K(R)) = K(0) = 0$.

On the other hand, if $K(R) \neq R$, then R has proper ideals of finite index. If $\bar{R} = R/K(R)$, then clearly $K(\bar{R})$ is the image of $K(R)$ in $\bar{R}$, that is, $K(\bar{R}) = 0$.

Thus, according to theorem 5, $K(R)$ behaves precisely like a radical should. We note that if R_n is the complete ring of $n\times n$-matrices over R, then $K(R_n)=K(R)_n$. Indeed, every ideal of finite index in R_n is of the form I_n, where I is an ideal of finite index in R.

Similiary, if $R=R_1\oplus R_2$, then $K(R)=K(R_1)\oplus K(R_2)$, since every ideal I of R is of the form $I=I_1\oplus I_2$, where I_i is an ideal of R_i. In particular, the class of rings R such that $K(R)=0$ is closed under formation of complete matrix rings, and direct sums.

If I is any ideal of R and $\bar{R}=R/I$, then $K(\bar{R})$ has as complete inverse image in R, the ideal $K_R(I)$ which is the intersection of all ideals J of finite index in R containing I. Cleaely $K(\bar{R})=0$ if and only if $I=K_R(I)$. We shall refer to such ideals as *K-radical ideals*. Notice that every ideal I of finite index is obviously a K-radical ideal.

The assumption that $K(R)=0$ in the case of left combinatorial rings is a reasonable simplification in the light of the following theorem.

THEOREM 6: *The ring R is left combinatorial if and only if $\bar{R}=R/K(R)$ is left combinatorial. Furthermore* $\lambda_R=\lambda_{\bar{R}}$ and $\lambda_R^*=\lambda_{\bar{R}}^*$.

Proof: If M is any finite R-module, then the annihilator of M is an ideal of finite index. Indeed, if $|M|=m$, and if I is this annihilator ideal, then M is a faithful R/I-module and the mapping from R/I to the left multiplications is an injection into M^M, i.e., $|R/I|\leq m^m$.

In particular, since $K(R)\subseteq I$, we may regard M as an $\bar{R}$-module with annihilator $I/K(R)$. The finite R-modules M_1 and M_2 ane R-isomorphic if and only if they are $\bar{R}$-isomorphic as $\bar{R}$-modules, since they have the same annihilator ideal containing $K(R)$ when they are considered as $\bar{R}$-modules. In particular this sets up a bijection between R-isomorphism classes and $\bar{R}$-isomorphism classes of modules M with $|M|=m$, i.e., $\lambda_R(m)=\lambda_{\bar{R}}(m)$ if one or the other is finite. Since this mapping also preserves decomposability it follows that $\lambda_R^*(m)=\lambda_{\bar{R}}^*(m)$ as well.

Some classes R_f identified

We divide the class of rings into two classes. R is of *class* 1 if R contains proper ideals of finite index, whence $R\neq K(R)$. Otherwise R is of *class* 2.

R is of class 1 if and only if R contains left ideals L of finite index. If R is of class 2 and if S is an epimorphic image of R, then S is of class 2, i.e., the collection of rings of class 2 is closed under epimorphic images. If $R=R_1\oplus R_2$, then R is of class 2 if and only if R_1 and R_2 are of classes 2. Furthermore, R_n is of class 2 if and only if R is of class 2. If R is of class 2, then R is trivially a combinatorial ring, since $\lambda_R(1)=1$ and if $m>1$, then $\lambda_R(m)=0$.

Furthermore, suppose that $\lambda_R(1)=1$ and that if $m>1$, then $\lambda_R(m)=0$. Then, if I is an ideal of finite index, it follows that $|R/I|=1$, i. e., $I=R$, whence $K(R)=R$ and R is of class 2.

We thus have rather easily:

THEOREM 7: *If f is the function with $f(1)=1$ and $f(m)=0$ otherwise, then R_f consists of the collection of rings of class 2, i. e., the collecticn of rings which do not contain proper ideals of finite index.*

Next we consider the class of rings R such that $\lambda_R(m)\leq 1$ for all positive integers m. Thus if $g(m)=1$ for all positive integers m, then the class of rings we are concerned with is $R_g=R_1$. If $R\in R_1$ we shall call R a *left-one* ring.

Obviously all divion rings are examples of left-one rings.

Now suppose R is a left-one ring let the sequence $\{m_1<m_2<\cdots\}$ be the sequence of integers $m\geq 1$ such that for all $i\geq 1$, there is an irreducible R-module M_i with $|M_i|=m_i$.

If M is any finite R-module, and if M is not irreducible and if N is a proper submodule of M, if follows that $M=N\oplus M/N$, since both are R-modules of the same cardinality. By continuing this process we may decompose $M=N_1\oplus\cdots\oplus N_k$ uniquely as a direct sum of (finite) irreducible submodules.

In particular, if R is finite and a left-one ring, then $R=I_1\oplus\cdots\oplus I_k$ is the direct sum of irreducible left ideals, whence R is a semi-simple Artinian ring. In particular $R=F_{1,n_1}\oplus\cdots\oplus F_{l,n_l}$, where F_{i,n_i} is the complete ring of $n_i\times n_i$-matrices over the field F_i.

If F is a (finite) field, then F admits only one irreducible module, viz., F. By theorem 2 it follows that F_n admits only one irreducible module, viz., M, consisting of all matrices X with $X_{ij}=0$ if $2\leq j\leq n$.

If $R=F_{1,n_1}\oplus\cdots\oplus F_{l,n_l}$, and if F_i contains q_i elements, then we have a sequence $\{m_1=q_1^{n_1}<m_2=q_2^{n_2}<\cdots<m_l=q_l^{n_l}\}$ without loss of generality. Now the numbers $\{m_1,\cdots,m_l\}$ must have the property that $m=m_1^{e_1}\cdots m_l^{e_l}=m_1^{f_1}\cdots m_l^{f_l}$ implies $e_1=f_1,\cdots,e_l=f_l$.

Indeed, in this case $\lambda_R(m)\leq 1$, since $M=M_1\oplus\cdots\oplus M_1\oplus\cdots\oplus M_l\oplus\cdots\oplus M_l$, where there are e_i copies of M_i, yields $|M|=m_1^{e_1}\cdots m_l^{e_l}$.

Obviously, this happens only if $i\neq j$ implies $(q_i, q_j)=1$, for otherwise $q_i=p^m$, $q_j=p^n$, will allow us to construct non-isomorphic modules M_1 and M_2 with $|M_1|=|M_2|=p^s$, where $s=am=bn$.

If R is an arbitrary left-one ring, then we consider $R/K(R)=\bar{R}$ as a left-one ring, i. e., we suppose that $K(R)=0$.

Now if I is a maximal ideal of finite index, then R/I is a finite left-one ring, and by the maximality of I, R/I is a complete matrix ring over a field, $R/I=(F_I)_{n(I)}$.

With R/I the associated irreducible R-module is M_I, the first columns of the matrices in $(F_I)_{n(I)}$, i.e., as an F_I-space $M_I=F_I^{n(I)}$, where if $|F_I|=q_I$, then $|M_I|=q_I^{n(I)}$.

Next if we consider $I=I_1\cap\cdots\cap I_k$, where each I_j is a maximal ideal of finite index, then $R/I=R/I_1\oplus\cdots\oplus R/I_k$, by observing that R/I is an R/I-module and a left-one ring, which can be decomposed as above.

If R is a left-one ring and I is any ideal of finite index, then considering M as an R/I-module, where $M=R/I$, we obtain $M=M_1\oplus\cdots\oplus M_k$, where each M_i is irreducible.

Since R/I is a finite left-one ring, it follows that $R/I=F_{1,n_1}\oplus\cdots\oplus F_{l,n_l}$ as above. Thus the M_i's are column-spaces in the matrix rings, and $I=I_1\cap\cdots\cap I_l$ where I_j is the inverse image of F_{j,n_j} in R, i.e., $R/I_j=F_{j,n_j}$. Hence I_j is an ideal which is maximal and of finite index.

Hence every ideal of finite index is the intersection of ideals of finite index which are maximal, and whose quotients are complete matrix rings over (finite) fields. It follows that if I and J are maximal ideals of finite index, and if R/I and R/J have characteristic p and q respectively, then $(p,q)=1$, since otherwise the unique factorization properties would be violated.

THEOREM 8: *A ring R is a left-one ring if and only if $R/K(R)$ is a subdirect product of complete matrix rings over finite fields of distinct characteristics containing the identity of the direct product.*

Proof: Suppose A is the collection of all maximal ideals of finite index of the left-one ring R. If $I\in A$, $R/I=(F_I)_n$, where F_I has characteristic p_I, say. Not let $R^*=\prod_{I\in A}(R/I)$, i.e., $R^*=\prod_{I\in A}(F_I)_{n_I}$.

Now map $R/K(R)=\bar{R}$ to R^* by ϕ, where $\phi(r+K(R))=\prod_{I\in A}(r+I)$. The kernel of this mapping in R is $\bigcap_{I\in A}I=K(R)$, whence ϕ is an injection.

Since $K(R)\subseteq I$, it follows that $\phi\cdot\pi_I:\bar{R}\to R/I$ is in fact an epimorphism for each $I\in A$, where $\pi_I:R^*\to R/I$ is the standard projection. Furthermore $\phi(1)$ is the identity of R^*. Conversely, if $R/K(R)$ is a subdirect product of matrix rings $(F_I)_{n_I}$, where I ranges over A, and where $I\neq J$ implies F_I and F_J have distinct characteristics, then if M is any finite irreducible R-module and if I' is the annihilator of M, with F_I the ring $\mathrm{Hom}_R(M,M)$, we have $R/I'=(F_I)_{n_I}$ in the usual way, with M a column-space of $(F_I)_{n_I}$.

In particular, if p_I is the characteristic of $M=M_I$, then $|M|=p_I^m$ for some m, and since there is no possible duplication, $\lambda_R(m)\leq 1$, i.e., R is a left-one ring as asserted. The theorem follows.

A left-one ring R of the type $K(R)=0$ may have ideals of 'infinite index'. Thus, let $R=\prod_p \mathbf{Z}/(p)$, where the index p ranges over all primes p. Also, if $I=\prod_{p\equiv 1(\mathrm{mod}4)} \mathbf{Z}/(p)$, then $R/I=\mathbf{Z}/(2)\oplus\prod_{p\equiv 3(\mathrm{mod}3)}\mathbf{Z}/(p)$, whence R/I is actually uncountable. We observe that by theorem 8, R is a commutative left-one ring.

Left combinatorial rings identified

Given a ring R, we shall call ideals I and J *congruent* if R/I and R/J are isomorphic as rings. Congruence is an equivalence relation on the set of ideals of R.

In this section we prove the following theorem.

THEOREM 9: *A ring R is left combinatorial if and only if the congruence classes of ideals of finite index are all finite.*

Proof: Suppose R is not left combinatorial. Let m be the smallest integer such that $\lambda_R(m)$ is not defined. Let $A_1, \cdots, A_k$ be the collection of non-isomorphic abelian groups A with $|A|=m$, i.e., $k=\lambda_{\mathbf{Z}}(m)$.

Let $\{M_i\}_{i=1}^{\infty}$ be an infinite collection of non-isomorphic R-modules with $|M_i|=m$. Then, without loss of generality, the M_i are all defined on the same abelian group A. Let I_i be the annihilator of M_i. Since M_i is a faithful R/I_i-module, we may regard R/I_i as a subset of A^A, using the identification of R/I_i with left multiplication maps.

Note that even if $I_1=I_2$, we may have R/I_1 and R/I_2 mapped onto the same subset of A^A in different ways or R/I_1 and R/I_2 may be mapped onto different subsets of A^A.

Since the maximum index of I_i is m^m, we may assume without loss of generality that all ideals I_i have the same index $k\leq m^m$.

Now, the number of ideals equal to I_1 in this circumstance is at most $k!\binom{m^m}{k}$, the number of ways to map a set of k (labelled) elements to a set of m^m elements.

Since this is finite, we may assume that no other ideal among the I_i is equal to I_1, and similarly for the other ideals. Thus, since the number of subsets with k elements is finite, we may suppose that all rings R/I_i are mapped to precisely the same subset of A^A. But this means that $R/I_1\cong R/I_2 \cong\cdots$, i.e., the ideals $\{I_i\}_{i=1}^{\infty}$ are all congruent and of the same finite index. Hence there is at least one infinite congruence class of ideals of finite index.

If $\{I_i\}_{i=1}^{\infty}$ is an infinite congruence class of ideals of finite index, then $\{M_i=R/I_i\}_{i=1}^{\infty}$ is an infinite collection of non-isomorphic R-modules with $|M_i| =m=|R/I_i|$ for each i. In particular, $\lambda_R(m)$ is not left combinatorial. The theorem follows.

COROLLARY 1: *A ring R is left combinatorial if and only if R is right combinatorial.*

Proof: This is obvious since the identifying condition involves only two-sided ideals.

From theorem 9 it is clear how one may construct a ring R which has an infinite number of indecomposable R-modules M all containing the same finite number of elements. Indeed, let F be a finite field and let $R=F^N$, where N is the set of positive integers with addition and multiplication component wise. The irreducible R-modules in question are $M_i=F_i$ obtained by taking all functions $f:N\to F$ with $f(j)=0$ if $j\neq i$.

Rings R with $K(R)=0$ identified

If $K(R)=0$, then 0 is the intersection of all ideals of finite index. Let $\{I_\lambda\}_{\lambda\in\Lambda}$ be this family of ideals. If $\phi:R\to\Pi_\lambda(R/I_\lambda)=R^*$ is given by $\phi(r)=\Pi_\lambda(r+I_\lambda)$, then ϕ is a monomorphism and $\phi(R)$ contains the identity of R.

Furthermore, if $\pi_\lambda:R^*\to R/I_\lambda$ is the projection epimorphism, then $\phi\circ\pi_\lambda$: $R\to R/I_\lambda$ is the natural epimorphism, whence R is a subdirect product of finite rings $\{R_\lambda=R/I_\lambda\}_{\lambda\in\Lambda}$.

Conversely, if $\{R_\lambda\}_{\lambda\in\Lambda}$ is a family of infinite rings and $\phi:R\to\Pi_\lambda R_\lambda=R^*$ a monomorphism such that $\phi(R)$ contains the identity of R^* and such that $\phi\circ\pi_\lambda:R\to R_\lambda$ is an epimorphism, where $\pi_\lambda:R^*\to R_\lambda$ is the standard projection, then $K(R)\subseteq\cap_\lambda I_\lambda$, where for all $\lambda\in\Lambda$, $I_\lambda=\phi^{-1}(Ker\pi_\lambda)=\phi^{-1}(Ker\pi_\lambda\cap\phi(R))$ is an ideal of finite index since $R_\lambda=R/I_\lambda$.

Since $x\in K(R)$ implies $\pi_\lambda\phi(x)=0$ and thus that $\phi(x)=0$, $K(R)=0$ since ϕ is a monomorphism.

We have therefore proven the following theorem.

THEOREM 10: *If R is n ring, then $K(R)=0$ if and only if R is a subdirect product of finite rings which contains the identity of the direct product of these rings.*

Remarks

Suppose F is a finite field and suppose R is an extension field of degree n over F. Suppose S is the complete ring of $n\times n$-matrices over F. Then $\lambda_R^*(m)=\lambda_S^*(m)$ for all m, since both R and S admit essentially one indecomosable (irreducible) module, viz., R itself and a minimal left ideal of S. From theorem 1 it follows that $\lambda_R=D\lambda_R^*=D\lambda_S^*=\lambda_S$, so that combinatorially R and S are equivalent, although they are not isomorphic. Indeed, R and S do not even contain the same number of elements.

On the other hand, R and S are not unrelated either.

If F_q denotes the finite field with q elements, let $R=\Pi_p \Pi_i F_{p^i}$, where p ranges over all primes and i ranges over all positive integers. Then $\lambda_R^*(p^i)=1$ and $\lambda_R^*(m)=0$ otherwise. Therefore $\lambda_R=\lambda_{\mathbf{Z}}$, so that again R and $\mathbf{Z}$ are combinatorially equivalent, although the rings are by no means isomorphic.

The problem of determining when precisely left-combinatorial rings R and S have $\lambda_R=\lambda_S$, has at least a solution in the case $\lambda_R(1)=1$, $\lambda_R(m)=0$ otherwise, as is described in theorem 7.

One may also derive general observations concerning this problem such as the following consequence of theorem 3. If $R=A\oplus B$, and if $\lambda_A=\lambda_B$, with $\lambda_R(m)\neq 0$ for some integer $m>1$, then $\lambda_A(m)=0$ for all $m>1$, that is, $A=K(A)$, whence $K(R)=K(B)\oplus A$ and $R/K(R)=B/K(B)$.

It seems reasonable therefore to investigate this problem, since it appears neither trivial nor totally impossible to handle.

The functions λ_R are multiplicative functions. Evidently this class of multiplicative functions is rather a large class, a subset of the class of all multiplicative functions. The question is open to determine conditions on a multiplicative function λ to guarantee that it is of the from $\lambda=\lambda_R$ for some ring R, and conversely to determine conditions on a multiplicative function λ which are necessary if $\lambda=\lambda_R$ for some ring R.

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