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Single Machine Sequencing With Random Processing Times and Random Deferral Costs

Sung H. Park*

ABSTRACT

A single machine stochastic scheduling problem is considered. Associated with each job is its random processing time and random deferral cost. The criterion is to order the jobs so as to minimize the sum of the deferral costs. The expected sum of the deferral costs is theroretically derived under the stochastic situation for each of several scheduling decision rules which are well known for the deterministic environment. It is also shown that certain stochastic problems can be reduced to equivalent deterministic problems. Two examples are illustrated to show the expected total deferral costs.

INTRODUCTION

Much of the literature on single machine sequencing considered the problems of scheduling a set of jobs involving fixed and known processing times, linear deferral costs (or waiting costs) and due-dates. However, in real life, the time taken to complete a job on a machine is almost invariably random and, frequently, the deferral cost of a job may not be completely specified with certainty.

Several authors [1, 2, 3, 6] have considered some stochastic situations on a single machine. Rothkoph [6] and Banerjee [1] discussed scheduling problems with random processing times; Crabill and Maxwell [3] studied the cases of random processing times and random due-dates; and a comprehensive study of stochastic environments is done by Conway et al. [2], in which a brief discussion of the case when the processing times and deferral costs are random can be found.

The problem to be discussed in this paper is that of scheduling n jobs on a single machine. Associated with each job is its random processing time and random deferral cost, and the optimization criterion is the minimization of the sum of the deferral costs of n jobs. We assume that all jobs are independent and available for processing at time zero, and setup times for the jobs are independent of job sequence and are included in processing times. Also we assume that the machine is continuously available from time zero until all jobs have been completed and that once processing begins on a job, it is processed to completion without interruption. For brevity probability density functin will be henceforth denotep by p. d. f., and cumulative distribution function will be denoted by c. d. f.,

*Department of Computer Science and Statistics, Seoul National University

The notation to be often used below is

P_i=a non-negative random variable representing the processing time of job i, $i=1, 2, \cdots, n$;

U_i=a non-negative random variable representing the deferral cost of fob i, $i=1, 2, \ldots, n$;

$g_i(p)$=the $p.d.f.$ of P_i;

$h_i(u)$=the $p.d.f.$ of U_i;

$G_i(p)$=the $c.d.f.$ of P_i;

$H_i(u)$=the $c.d.f.$ of U_i;

(i)=the job scheduled to be drocsssed in the i^{th} position in sequence. For emample, if jobs 1, 2, 3 are scheduled in the order 3, 1, 2, then $(1)=3$, $(2)=1$, and $(3)=2$.

$F_{(i)}$=the flow-time (or completion time) of the job in i^{th} position of an an arbitrary sequence, i.e., $F_{(i)}=\Sigma^i{}_{j=1} P_{(j)}$.

If job i is completed at time F_i, then its deferral cost is U_iF_i and the total deferral cost of a schedule is given by

$$C=\sum_{1=1}^{n} U_{(i)}F_{(i)}=\sum_{i=1}^{n} U_{(i)}\sum_{j=1}^{i} P_{(j)} \text{ or } \sum_{i=1}^{n}\sum_{j=1}^{i} U_{(i)}P_{(j)}. \quad (1)$$

Obviously the total deferral cost C is a random variable and, before the execution of the jobs, C is unknown. When both processing times and deferral costs are deterministic and known, it was proved by Smith [8] that the total deferral cost is minimized by scheduling the jobs in order of increasing ratio of P_i/U_i. Rothkoph [6] extends this result to the case when the processing times are random and the deferral costs are known. He shows that the expected total deferral cost is minimized by scheduling the jobs in increasing order of $E(P_i)/U_i$ where $E(P_i)$ means th expected value of P_i.

The main purpose of this paper is to theoretically derive the expected total deferral costs under several different situations discussed below when the processing times and the deferral costs are random. For convenience, the following work is divided into two parts. The first part is a special case in that the p_i are independent identically distributed (will be referred to $i, i, d.$) random variables from a common $p.d.f.$ $g(p)$ and, similarly, the U_i are from a common $p.d.f.$ $h(u)$. The second part is the general case in that each job i draws its P_i and U_i from its own $g_i(p)$ and $h_i(u)$, respectively.

FROM COMMON PROBABILITY DENSITY FUNCTIONS

Suppose that the P_i are $i.i.d.$ random variables from $p.d.f.$ $g(p)$ and the U_i are $i.i.d.$ random variables from $p.d.f.$ $h(u)$. We also assume that the P_i and U_i are non-negative and statistically indepent. Then $P_1, P_2, \ldots, P_n$ may be regarded as a random sample of size n from a distribution of the random variable p having $p.d.f.$ $g(p)$ and $U_1, U_2, \ldots, U_n$ as a random sample from a distribution of the random variable U whose $p.d.f.$ is $h(u)$.

Suppose the jobs are arranged in the following way they are processed in that order:

1) random sequencing,
2) $P_{(1)} \leq P_{(2)} \leq \ldots \leq P_{(n)}$,
3) $U_{(1)} \geq U_{(2)} \geq \ldots \geq U_{(n)}$,

Since a useful standard of comparison may be provided by random sequencing of the jobs, random sequencing is also considered.

Let $E(C_i)$, $i=1,2,3$ and 4, denote the expected total deferral cost for the i^{th} case listed above. First of all, if the jobs are sequenced at random, it is readily shown that

$$E(C_1)=E(\sum_{i=1}^{n} U(i)\sum_{i=1}^{n} P_{(j)})$$

$$=E(U)E(P)\sum_{i=1}^{n} i$$

$$=n(n+1)E(U)E(P)/2, \qquad (2)$$

since the jobs are assigned to a position in sequence in a manner that does not depend on the values of processing time and deferral cost. A similar result can also be found in Conway et al. [2].

Secondly, if the jobs are arranged so that $P_{(1)}\leq P_{(2)}\leq\ldots\leq P_{(n)}$, then

$$E(C_2)=E(\sum_{i=1}^{n} U_{(i)}\sum_{j=1}^{i} P_{(j)})$$

$$=E(U)\sum_{i=1}^{n}(n-i+1)E(P_{(i)}). \qquad (3)$$

Note that $P_{(i)}$ is i^{th} order statistic of the random sample $P_1, P_2, \ldots, P_n$. From the theory of order statistics (for a reference see Hogg and Craig [4]), the $p.d.f.$ of the i^{th} order statistic for the processing time is given by

$$S_i(P)=\frac{n!}{(i-1)!(n-1)!}[G(p)]^{i-1}[1-G(p)]^{n-i}g(p),$$

where $F(p)$ is the $c.d.f.$ of the processing time p. The expected value of the i^{th} ordered processing time is

$$E(p_{(i)})=\int_0^{\infty} p s_i(p)\,dp.$$

Since it is not immediately clear that $E(C_2)$ is less than $E(C_1)$, we obtain the second term in (3) in a simpler form,

$$\sum_{i=1}^{n}(n-i+1)E(P_{(i)})$$

$$=\sum_{i=1}^{n}(n-i+1)\int_0^{\infty}\frac{pn!}{(i-1)!(n-i)!}[G(p)]^{i-1}[1-G(p)]^{n-i}g(p)\,dp$$

$$=\int_0^{\infty} p\sum_{i=1}^{n}\frac{(n-i+1)n!}{(i-1)!(n-i)!}[G(p)]^{i-1}[1-G(p)]^{n-i}g(p)\,dp$$

$$=\int_0^{\infty} np\sum_{j=0}^{n-1}\frac{((n-j-1)+1)(n-1)!}{j!(n-j-1)!}[G(p)]^{j}[1-G(p)]^{n-j-1}g(p)\,dp$$

$$=\int_0^{\infty} np[(n-1)(1-G(p))+1]g(p)\,dp$$

$$=n[nE(p)-(n-1)W_p] \qquad (4)$$

where $W_p=\int_0^{\infty} pG(p)g(p)\,dp$. Substituting (4) into (3), the expected total deferral cost may be written as

$$E(C_2)=E(U)[n(nE(P)-(n-1)W_p]$$
$$=n(n+1)E(U)E(P)/2-n(n-1)E(U)(W_p-E(P)/2). \quad (5)$$

The second term on the right in (5) is always positive, since it can be shown (for proof see Appendix 1) that $W_p > E(P)/2$ for any non-degenerate *p.d.f.* g(p). Therefore, it is clear that taking the jobs in order of increasing processing times will result in a smaller total deferral cost than would be obtained in a random sequence.

Thirdly, if the jobs are arranged so that $U_{(1)} \geq U_{(2)} \geq \ldots \geq U_{(n)}$, the expected total deferral cost may be found by a similar argument made in (2) through (5). The complete derivation is not shown here, but one can show that

$$E(C_3)=E\left(\sum_{i=1}^{n} U_{(i)} \sum_{j=1}^{i} P_{(j)}\right)$$
$$=E(P)E\left(\sum_{i=1}^{n} iU_{(i)}\right)$$
$$=n(n+1)E(U)E(P)/2-n(n-1)E(P)(W_u-E(U)/2), \quad (6)$$

where $W_u=\int_0^\infty uH(u)h(u)du$, and $U_{(i)}$ is the $(n-i+1)^{th}$ order statistic of the random sample $U_1, U_2, \ldots, U_n$. Comparing $E(C_1)$ and $E(C_3)$, it is obvious that taking the jobs in order of decreasing deferral costs will, in general, result in a smaller total deferral cost than would be obtained in a random sequence. However, there is not clear-cut choice between $E(C_2)$ and $E(C_3)$, since the difference $E(C_2)-E(C_3)$ can take on a positive or negative value depending on the *p.d.f.* $g(p)$ and $h(u)$. If $g(p)$ and $h(u)$ are given, of course, it is possible to find the better choice.

Suppose the jobs are sequenced so that

$$P_{(1)}/U_{(1)} \leq P_{(2)}/U_{(2)} \leq \ldots \leq P_{(n)}/U_{(n)}. \quad (7)$$

It would be much more difficult to derive the expected total deferral cost for this case, because $P_{(i)}$ is not necessarily the i^{th} ordered processing time and $U_{(i)}$ is not necessarily the $(n-i+1)^{th}$ ordered deferral cost, although $P_{(i)}/U_{(i)}$ is the i^{th} order statistic of the random sample: $P_1/U_1, P_2/U_2, \ldots, P_n/U_n$.

EXAMPLE I

Suppose we have a five job sequencing problem on a single machine. It is assumed that the *p.d.f.* of the processing time P is a Gamma distribution with mean 3 minutes,

$$g(p)=p^2e^{-p}/2, \quad p \geq 0,$$

and the *p.d.f* of the deferral cost U is also a Gamma distribution with mean \$4/min,

$$h(u)=u^3e^{-u}/6, \quad u \geq 0.$$

The expected total deferral cost for a random sequence is from (2)

$$E(C_1)=n(n+1)E(U)E(P)/2=\$180$$

since $n=5$, $E(U)=4$ and $E(P)=3$,

In order to show $E(C_2)$ and $E(C_3)$, the expected values of order statistics, $P_{(j)}$ and $U_{(j)}$, are obtained from Sarah and Greenberg [7]. They are

$E(P_{(1)})=1.321 \qquad E(U_{(1)})=6.521$

$E(P_{(2)})=2.045 \qquad E(U_{(2)})=4.803$

$E(P_{(3)})=2.768 \qquad E(U_{(3)})=3.767$

$E(P_{(4)})=3.668 \qquad E(U_{(4)})=2.905$

$E(P_{(5)})=5.197 \qquad E(U_{(5)})=2.002.$

Therefore, from (3) and (6) it can be shown that

$$E(C_2)=E(U)\sum_{i=1}^{n}(n-i+1)E(P_{(i)})=\$142.49$$

and

$$E(C_3)=E(P)E(\sum_{i=1}^{n} iU_{(i)})=\$147.19.$$

FROM SEPARATE PROBABILITY DENSITY FUNCTIONS

The model considered in this section is that each iob i draws its processing time P_i from separate $g_i(p)$ and its deferral cost U_i from $h_i(u)$. It is also assumed that the processing times and the deferral costs are stochastically independent. Since each job may have a different *a priori* estimate of its processing time, deferral cost or ratio P_i/U_i, a scheduling decision can be made based on the *a priori* estimates. This is not an unrealistic situation, for one often uses the past information for similar jobs when the actual processing time or deferral cost of a specific job is not known.

Suppose that $E(P_i)$, $E(U_i)$ and $E(P_i/U_i)$ for a l l i are known and that the jobs are sequenced so that

1) random sequencing,
2) $E(P_{(1)})\leq E(P_{(2)}) \geq \ldots \leq E(P_{(n)})$,
3) $E(U_{(1)})\geq E(U_{(2)})\geq \ldots \geq E(U_{(n)})$,
4) $E(P_{(1)}/U_{(1)})\leq E(P_{(2)}/U_{(2)})\leq \ldots E(P_{(n)}/U_{(n)})$

or

5) $E(P_{(1)})/E(U_{(1)})\leq E(P_{(2)})/E(U_{(2)})\leq \ldots \leq E(P_{(n)}/E(U_{(n)})$.

Let $F(P_1, P_2, \ldots, P_n, u_1, u_2, \ldots u_n)$ be the joint $c.d.f.$ of the P_i and U_i. The the total deferral cost of an arbitrary sequence is given by

$$TC=\sum_{i=1}^{n} U_{(i)}\sum_{j=1}^{i} P_{(j)},$$

and the expected value of the total deferral cost would be

$$E(TC)=\int_0^\infty\cdots\int_0^\infty\sum_{i=1}^{n}\sum_{j=1}^{i} U_{(i)}P_{(j)}dF(p_1, p_2, \ldots, p_n, u_1, u_2, \ldots u_n)$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{i}\int_0^\infty\cdots\int_0^\infty U_{(i)}P_{(j)}dF(p_1, \;_2, \ldots, p_n, u_1, u_2, \ldots, u_n)$$

$$=\sum_{i}^{n}\sum^{i} E_{(i)}P_{(j)})$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{i} E(U_{(i)})E(P_{(j)}) \text{ or } \sum_{i=1}^{n} E(U_{(i)})\sum_{j=1}^{i} E(P_{(j)}), \tag{8}$$

where the equality in (8) comes from the fact that the $P_{(j)}$ and $U_{(i)}$ are stochastically independent.

Let $E(TC_i)$, $i=1, 2, \ldots, 5$, denote the expected total deferral cost for the i^{th} scheduling decision listed above. First of all, consider the case that the jobs are sequenced completely at random. Since position i is equally likely to be occupied by any one of the n jobs,

$$E(U_{(i)})=E(\sum_{i=1}^{n} U_i)/n=\sum_{i=1}^{n} E(U_i)/n=u^*$$

and

$$E(P_{(i)})=E(\sum_{i=1}^{n} P_i)/n=\sum_{i=1}^{n} E(P_i)/n=p^*,$$

for $i=1, 2, \ldots, n$. Therefore. the expected total deferral cost for random sequencing will be

$$E(TC_1)=\sum_{i=1}^{n}\sum_{j=1}^{i} E(U_{(i)})E(P_{(j)})$$

$$=u^*\sum_{j=1}^{n}(n-j+1)E(P_{(j)}), \tag{9}$$

where $E(P_{(j)})$ is the expected value of the processing time P_i of job i whose position in sequence is j, *i.e.*, $(j)=i$.

Thirdly, if the jobs are processed in order of decreasing expected deferral costs, then

$$E(TC_3)=\sum_{i=1}^{n}\sum_{j=1}^{i} E(U_{(i)})E(P_{(j)})$$

$$=p^*\sum_{i=1}^{n} iE(U_{(i)}). \tag{10}$$

Fourth, if the jobs are sequenced in increasing order of $E(P_i/U_i)$, then

$$E(TC_4)=\sum_{i=1}^{n}\sum_{j=1}^{i} E(U_{(i)})E(P_{(j)}), \tag{11}$$

where $E(U_{(i)})$ and $E(P_{(i)})$ are the expected values of the P_j and U_j of job j whose position in sequence is i, *i.e.*, $(i)=j$.

Fifthly, if the jobs are processed in increasing order of $E(P_i)/E(U_i)$, then $E(TC_5)$ has the same form as $E(TC_4)$ in (11). However, the order of jobs provided by the 5^{th} scheduling scheme is not necessarily equal to that provided by the 4^{th} one, since $E(P_i/U_i)$ is not exactly equal to $E(P_i)/E(U_i)$. In fact, $E(P_i/U_i)$ is slightly greater than $E(P_i)/E(U_i)$. For this discussion see Appendix 2. But, for many probability distributions commonly used in real life, it is unlikely that the both schduling decisions would propose different sequences, mainly because a consistent bias in the estimates of $E(P_i/U_i)$ for all i would have no significant effect on the order of jobs.

We have studied the expected total deferral costs for various scheduling schemes. Which scheduling rule leads to the optimal schedule in the stochastic situation? That is, which one among $E(TC_i)$, $i=1, 2, \ldots, 5$, is the minimum? Since, in the deterministic case the total deferral cost,

$$TC=\sum_{i=1}^{n}\sum_{j=1}^{i} U_{(i)}P_{(j)},$$

would be minimized by processing the jobs in increasing $P_{(i)}/U_{(i)}$ order as shown by Smith [8], processing the jobs in increasing orer of $E(P_{(i)})/E(U_{(i)})$ clearly minimize

$$E(TC)=\sum_{i=1}^{n}\sum_{j=1}^{i} E(U_{(i)})E(P_{(j)}).$$

This is a simple, but significant result, since it is the probabilitic counterpart of the optimal rule in the deterministic situation. Note that it reduces the stochastic problem to the equivalent deterministic problem.

An important observation to be made is that, if the P_i and U_i are not stochastically independent, the equality in (8) does not hold true. and $E(TC)$ in (8) should be written as

$$E(TC)=E(\sum_{i=1}^{n}\sum_{j=1}^{i} U_{(i)}P_{(j)}).$$

Since $(\sum_{i=1}^{n}\sum_{j=1}^{i} U_{(i)}P_{(j)})$ is minimized by processing the jobs in increasing $P_{(i)}/U_{(i)}$ order, a probabilistic consequence is that $E(\sum_{i=1}^{n}\sum_{j=1}^{i} U_{(i)}P_{(j)})$ would be minimized if the jobs are sequenced in increasing order of $E(P_{(i)}/U_{(i)})$.

EXAMPLE II

Suppose we have 3 jobs whose processing times and deferral costs are Gamma-distributed,

$$G(x|\alpha,\beta)=\frac{\alpha^{\beta}}{\gamma(\beta)}e^{-\alpha x}x^{\beta-1},\quad \alpha,\beta,\ x\geq 0. \tag{12}$$

For brevity, assume $\alpha=1/2$ for the disributions of all P_i and U_i, The β values are shown in Table 1.

	β values	
	P_i (minutes)	U_i ($/min)
job 1	9	10
job 2	5	9
job 3	7	14

Table 1

Since $E(x)$ for the Gamma variable in (12) is $E(x)=\beta/\alpha$, it can be easily shown that

$E(P_1)=18,\ E(P_2)=10,\ E(P_3)=14,$

$E(U_1)=20,\ E(U_2)=18,\ E(U_3)=28.$

In general, if X_1 is a Gamma variable with p.d.f. $G(x_1|\alpha,\beta_1)$ and X_2 is another Gamma variable with $G(x_2|\alpha,\beta_2)$, then the p.d.f. of the ratio of two independent Gamma variables, $R=X_1/X_2$, is (see Rao [5])

$$f(r|\beta_1, \beta_2) = \frac{\Gamma(\beta_1+\beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \cdot \frac{r^{\beta_1-1}}{(1+r)^{\beta_1+\beta_1}}$$

with $E(R) = \beta_1/(\beta_2-1)$. Therefore,

$E(P_1/U_1) = 1$, $E(P_2/U_2) = 5/8$ and $E(P_3/U_3) = 7/13$.

The expected total deferral cost for random sequencing in (9) will be

$E(TC_1) = n(n+1)u^*p^*/2 = \$1,848$

since $n=3$, $u^*=22$ and $p^*=14$. If the jobs are sequenced in order of increasing expected processing times, then (1) $=2$, (2) $=3$ and (3) $=1$ with

$$\mathrm{E}(TC_2) = u^* \sum_{j=1}^{n} (n-j+1) E(P_{(j)}) = \$1,672.$$

If the jobs are proessed in order of decreasing expected deferral costs, then (1) $=3$, (2) $=1$ and (3) $=2$ with

$$E(TC_3) = p^* \sum_{i=1}^{n} iE(U_{(i)}) = \$1,708.$$

Lastly, if the jobs are sequenced in increasing order of $E(P_i/U_i)$ or $E(P_i)/E(U_i)$, then (1) $=3$, (2) $=2$ and (3) $=1$, and the expected total deferral cost would be

$$E(TC_4 \text{ or } TC_5) = \sum_{i=1}^{n} \sum_{j=1}^{i} E(U_{(i)}) E(P_{(j)}) = \$1,664.$$

APPENDIX 1

We wan to show that $W_p \geq E(p)/2$, that is

$$\int_0^\infty pG(p)g(p)\,dp \geq 1/2 \int_0^\infty pg(p)\,dp. \quad (1)$$

Let $a = \mathrm{Sup}\{p; G(p)=0\}$ and $b = \inf\{p: G(p)=1\}$. Then the function $G:[a,b] \to [0,1]$ has an inverse, say $M:[0,1] \to [a,b]$. Now in equation (1), change variables according to $y=G(p)$ or $p=M(y)$. Note that $G(p)$ and $M(y)$ are increasing functions. Then

$$\int_a^b pG(p)g(p)\,dp = \int_0^1 M(y)\,y\,dy,$$

and $1/2\int_a^b pg(p)\,dp = 1/2\int_0^1 M(y)\,dy$.

So we want to prove

$$\int_0^1 M(y)(y-1/2)\,dy \geq 0. \quad (2)$$

Now $\int_0^1 = \int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^1$. In the first integral put $y-1/2 = -Z$; in the second, $y-1/2=Z$. Thus inequality (2) reduces to

$$\int_0^{\frac{1}{2}} [M(1/2+Z) - M(1/2-Z)]Z\,dZ \geq 0. \quad (3)$$

The truth of inequality (3) is self-evident.

APPENDIX 2

We want to show $E(x/y)$ for two non-negative and independent random variables x and

y. Let $E(x)=x^*$ and $E(y)=y^*$. By a Taylor series expansion in two variables, x/y may be expressed as

$$x/y=x^*y^*+(x-x^*)/y^*-x^*(y-y^*)/(y^*)^2-(x-x^*)(y-y^*)/(y^*)^2$$
$$+x^*(y-y^*)^2/(y^*)^3+(x-x^*)(y-y^*)^2/(y^*)^3-x^*(y-y^*)^3/(y^*)^4+\ldots.$$

Taking expectations, we can obtain

$$E(x/y)=x^*/y^*+x^*E(y-y^*)^2/(y^*)^3-x^*E(y-y^*)^3/(y^*)^4+\ldots,$$
$$=x^*/y^*+(x^*/y^*)E\left[\sum_{i=1}^{\infty}(-1)^i(y-y^*)^i/(y^*)^i\right]$$
$$=(x^*/y^*)E\left[\sum_{i=0}^{\infty}((y^*-y)/y^*)^i\right]$$
$$=(x^*/y^*)E[1/(1-(y^*-y)/y^*]$$
$$=(x^*/y^*)E(y^*/y)$$
$$=x^*/y^*+x^*[E(1/y)-1/y^*].$$

From Jensen's inequality $E(1/y)\geq 1/y^*$ (with equality only for a degenerate distribution of y), which means that $E(x/y)$ is slightly greater than x^*/y^* for a non-degenerate distribution of y.

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