Latin Square Type Partially Balanced Incomplete Block Designs

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ABSTRACT

It is well known that $L_2(m)$ type PBIB designs have the Property A, so they are BNAS PBIB designs. However, $L_3(m)$ type PBIB designs are not of type of Property A but do have the factorial structure (Cotter, John, and Smith(1973)). In this paper, the properties of the $L_3(m)$ type PBIB designs are investigated. Extended Property A and fractional BNAS are defined and a solution formula for the treatment effects in the $L_3(m)$ type designs is obtained.

1. Introduction

Partially Balanced Incomplete Block(PBIB) designs of the Latin square type were introduced by Bose and Shimamoto(1952). They were tabled in a monograph by Bose, Clatworthy, and Shrikhande(1954). The symbol $L_g(m)$ is to designate a Latin square type of $v=m^2$ treatments by the following definition.

In a PBIB design with parameters $(v=m^2, r, k, b;\ \lambda_1, \lambda_2)$, suppose it is possible to form a square array of m rows and m columns filled with the treatment numbers $1, 2, \cdots, m^2$, so that two treatments are first associates if they occur in the same row or same column of the array and are second associates otherwise. Such a design will be said to belong to the sub-type $L_2(m)$ of the Latin square type design. We also have designs with m^2 treatments belonging to the sub-type $L_3(m)$ of the Latin square type design. In this case, it is possible to form a

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square array of m^2 numbers and to impose a Latin square with m letters on this array, so that any two treatments are first associates if they occur in the same row or column of the array or correspond to the same letter and are second associates otherwise.

It is well known that $L_2(m)$ type PBIB designs have the Property A which was introduced by Kurkjian and Zelen(1963), but $L_3(m)$ type PBIB designs are not of type of Property A. In this paper, the properties of $L_3(m)$ type PBIB designs are investigated. Extended Property A and Fractional BNAS are defined and they are used to obtain a solution for the treatment effects.

2. Preliminaries

The following notations(after Kurkjian and Zelen(1963)) will be used:

$\mathbf{1}_{m_i}=m_i\times 1$ column vector having all elements unity,

$\boldsymbol{J}_{m_i}=m_i\times m_i$ matrix with all elements unity,

$\boldsymbol{I}_{m_i}=m_i\times m_i$ identity matrix; $\boldsymbol{I}_i^{x_i}=\mathbf{1}_{m_i}$ if $x_i=0$ and $\boldsymbol{I}_{m_i}$ if $x_i=1$,

$\boldsymbol{D}_i^{\delta_i}=\boldsymbol{I}_{m_i}$ if $\delta_i=0$ and $\boldsymbol{J}_{m_i}$ if $\delta_i=1$,

$\boldsymbol{M}_i=m_i\boldsymbol{I}_{m_i}-\boldsymbol{J}_{m_i}$, $\boldsymbol{M}_i^{x_i}=\mathbf{1}_{m_i}'$ if $x_i=0$ and $\boldsymbol{M}_i$ if $x_i=1$.

The Kronecker product of $\boldsymbol{M}_i$ and $\boldsymbol{M}_j$ will be written as $\boldsymbol{M}_i\otimes\boldsymbol{M}_j$ and in general, the joint Kronecker product of $n\boldsymbol{M}_i(i=1,2,\cdots,n)$ will be written as $\prod_{i=1}^{n}\otimes\boldsymbol{M}_i$.

Let v teatments be assigned to b blocks of k plots each in such a way that each treatment is replicated r times and ith treatment occurs $n_{ij}(n_{ij}=0$ or $1)$ times in the jth block. The $v\times b$ matrix $(n_{ij})=\boldsymbol{N}$ is the incidence matrix of the design. The usual additive set-up with normal indepedent errors with a common variance σ^2 is assumed. The reduced normal equations for estimating the treatment effects are known to be

$$\boldsymbol{C}\hat{\boldsymbol{t}}=\boldsymbol{Q}, \tag{2.1}$$

where $\boldsymbol{Q}=\boldsymbol{T}-\frac{1}{k}\boldsymbol{NB}$; $\boldsymbol{T}$=column vector of the v treatment totals; $\boldsymbol{B}$=column vector of the b block totals; $\boldsymbol{t}$=column vector of the v treatment effects; and

$$C=rI_v-\frac{1}{k}NN'. \tag{2.2}$$

A solution to the equation (2. 1) is given by

$$\hat{t}=C^{-1}Q, \tag{2.3}$$

where C^{-1} is a generalized inverse of C, that is, satisfies $CC^{-1}C=C$.

Kurkjian and Zelen(1963) introduced a structural property of the design matrix NN' with $v=\prod_{i=1}^{n}m_i$. A block design will be said to have Property A if

$$NN'=\sum_{s=0}^{n}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} h(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}\otimes D_i^{\delta_i}\right\}, \tag{2.4}$$

where $\delta_i=0$ or 1 for $i=1,2,\cdots,n$ and $h(\delta_1,\delta_2,\cdots,\delta_n)$ are constants. In this case, we obtain the following C^{-1} matrix for the equation (2. 3):

$$C^{-1}=\sum_{s=1}^{n}\left\{\sum_{x_1+x_2+\cdots+x_n=s}\frac{\prod_{i=1}^{n}\otimes I_i^{x^i}M_i^{x^i}}{vr\theta(x_1,x_2,\cdots,x_n)}\right\} \tag{2.5}$$

where

$$r\theta(x_1,x_2,\cdots,x_n)=\sum_{s=0}^{n-1}\left\{\sum_{\delta_1+\delta_2+\cdots+\delta_n=s} g(\delta_1,\delta_2,\cdots,\delta_n)\prod_{i=1}^{n}m_i^{(1-x_i)\delta_i}(1-x_i\delta_i)\right\}$$

for $g(0,0,\cdots,0),=r-\frac{1}{k}h(0,0,\cdots,0)$, $g(\delta_1,\delta_2,\cdots,\delta_n)=-\frac{1}{k}h(\delta_1,\delta_2,\cdots,\delta_n)$ if $(\delta_1,\delta_2,\cdots\delta_n)\neq 0$.

In the case of $v=\prod_{i=1}^{n}m_i$, the ith treatment can be denoted by the n-tuple, $i=(i_1,i_2,\cdots,i_n)$, $i_j=0,1,\cdots,m_j-1$, and the treatments are written in lexicographical order, then the two treatments are the $(p_1,p_2,\cdots,p_n)$th associates, while $p_i=1$, if the i th factor occurs at the same level in both treatments and $p_i=0$ otherwise: $\lambda_{p_1p_2\cdots p_n}$ will denote the number of times these treatments occur together in a block. The association scheme may be called a Binary Number Association Scheme(BNAS).

Paik and Federer(1973) showed that every Property A Type Incomplete Block design is a PBIB design with BNAS, and conversely.

3. Properties of $L_3(m)$ type PBIB

It can be easily verified that $L_2(m)$ type PBIB is a PBIB having BNAS such that $v=m^2$, $\lambda_{10}=\lambda_{01}, \lambda_{00}, \lambda_{11}=r$, and it has the following Property A:

$$NN'=(r+\lambda_2-\lambda_1)I_m\otimes I_m-(\lambda_2-\lambda_1)(I_m\otimes J_m+J_m\otimes I_m)+\lambda_2 J_m\otimes J_m \quad (3.1)$$

where $\lambda_1=\lambda_{10}=\lambda_{01}$, $\lambda_2=\lambda_{00}$, and assume $\lambda_1<\lambda_2$.

In the case of $L_3(m)$ type PBIB, however, it is not of type of Property A nor of PBIB having BNAS. In an $L_3(m)$ type PBIB, the treatments can be written as a $1/m$ fraction of an m^3-factorial and in this case, the following association forms will be obtained: (0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0), and (1, 1, 1). So, we may call this "Fractional BNAS", and we see that

$$\lambda_1=\lambda_{001}=\lambda_{010}=\lambda_{100},\ \lambda_2=\lambda_{000},\ \text{and}\ r=\lambda_{111}.$$

In the matrix NN' given an $L_3(m)$ type PBIB design, put $\lambda_{111}=\lambda_{001}=1$ and $\lambda_{010}=\lambda_{100}=\lambda_{000}=0$, then we obtain an $m^2\times m^2$ matrix L with the following properties:

$$L=(L_{ij}),\ i=1, 2, \cdots, m;\ j=1, 2, \cdots, m,$$

where L_{ij} is an $m\times m$ matrix exactly one unit element in every row and column, all other elements zero and the pattern of the matrix L_{ij} depends upon the subscripts i and j and the form of the Latin square. Also we have

$$\begin{aligned}&\sum_i L_{ij}=\sum_j L_{ij}=J_m,\\ &L_{ij}J_m=J_mL_{ij}=J_m \text{ for all } i \text{ and } j,\\ &LL=mL \text{ and } LJ_{mm}=J_{mm}L=mJ_{mm}.\end{aligned} \quad (3.2)$$

Now, design matrix NN' of $L_3(m)$ type PBIB is following form (This property of NN' could be called "Extended Property A"):

$$NN'=(r+2\lambda_2-\lambda_1)I_m\otimes I_m-(\lambda_2-\lambda_1)(I_m\otimes J_m+J_m\otimes I_m)+\lambda_2 J_m\otimes J_m-(\lambda_2-\lambda_1)L \quad (3.3)$$

Therefore,

$$C=\frac{r(k-1)-2\lambda_2-\lambda_1}{k}I_{mm}+\frac{\lambda_2-\lambda_1}{k}(I_m\otimes J_m+J_m\otimes I_m)-\frac{\lambda_2}{k}J_{mm}+\frac{\lambda_2-\lambda_1}{k}L \quad (3.4)$$

Using (3.2), we obtain the generalized inverse of C as follows:

$$C^{-}=\frac{k}{m^2[r(k-1)+(m-2)\lambda_2+(1-m)\lambda_1]}\;[I_1{}^0M_1{}^0\otimes I_2{}^1M_2{}^1+I_1{}^1M_1{}^1\otimes I_2{}^0M_2{}^0]$$
$$+\frac{k}{m^2[r(k-1)-2\lambda_2+\lambda_1]}I_1{}^1M_1{}^1\otimes I_2{}^1M_2{}^1+aL$$
$$=\frac{k}{m^2[r(k-1)+(m-2)\lambda_2+(1-m)\lambda_1]}\;[J_m\otimes(mI_m-J_m)+(mI_m-J_m)\otimes J_m]$$
$$+\frac{1}{m^2[r(k-1)-2\lambda_2+\lambda_1]}[(mI_m-J_m)\otimes(mI_m-J_m)]+aL, \tag{3.5}$$

where

$$a=\frac{-k(\lambda_2-\lambda_1)}{[r(k-1)-2\lambda_2+\lambda_1][r(k-1)+(m-2)\lambda_2+(1-m)\lambda_1]} \tag{3.6}$$

4. Example

$v=4\times4=16,\ \ r=3,\ \ k=3,\ \ b=16,\ \ \lambda_1=0,\ \ \lambda_2=1$

Association scheme of $L_3(4)$

1A	2B	3C	4D
5B	6A	7D	8C
9C	10D	11B	12A
13D	14C	15A	16B

Plan

blocks	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯
	1	1	1	2	2	2	3	3	3	4	4	4	5	6	7	8
	7	8	11	7	8	12	5	6	10	5	6	9	10	9	12	11
	8	10	14	9	15	13	12	13	16	14	11	15	15	16	14	13

$NN'=5I_4\otimes I_4-I_4\otimes J_4-J_4\otimes I_4+J_4\otimes J_4-L$

where $L=(L_{ij}),\ i=1,2,3,4;\ j=1,2,3,4,$ and

$$L_{11}=L_{22}=L_{33}=L_{44}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix},\quad L_{12}=L_{21}=L_{34}=L_{43}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

$$L_{13}=L_{24}=L_{32}=L_{41}=\begin{pmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{pmatrix},\quad L_{14}=L_{23}=L_{31}=L_{42}=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&1&0&0\\1&0&0&0\end{pmatrix}$$

In this case,

$$C=\frac{1}{3}[4I_{16}+I_4\otimes J_4+J_4\otimes I_4-J_{16}+L]$$

From (3.5) and(3.6), we obtain

$$C^-=\frac{3}{32}[8I_{16}-I_4\otimes J_4-J_4\otimes I_4-L]$$

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