A Linear and Consistent Class of Econometric Estimators in Simultaneous Equations

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1. Introduction

Straight-forward application of the ordinary least squares method for estimating the parameters of a simultaneous linear stochastic equations model does not provide consistent estimators due to the fact that the explanatory jointly dependent variables are correlated with the disturbances. The search for consistent estimators during the last three decades has yielded a variety of estimators which can be broadly classified into two groups, namely, limited information and full information. Both the groups recognize the dependence of explanatory variables but the former group fails to utilize the over-identifying restrictions in the structural equations except the one under study while the latter group succeeds; see, e.g. Srivastava(1978) for a brief review and Theil (1961) for a detail description.

Assuming disturbances to be normally distributed, Hendry(1976) demonstrated that all the well-known limited information and full information estimators can be obtained from an estimator generating equation stemming from

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the full information maximum likelihood method and can be interpreted as numerical approximations to the full information maximum likelihood estimator. This paper develops a class of linear and consistent estimators without assuming normality of disturbances, and it is shown that all the known econometric estimators are members of this class. This presents a unified treatment of econometric estimators and may provide a fresh motivation for developing possibly different and efficient estimators and exploration for the relationships among the existing estimators.

2. The Linear and Consistent Class

Consider a complete system of M linear structural equations in M jointly dependent and Λ predetermined variables:

$$YC+XB=U \tag{2.1}$$

where Y is a $T\times M$ matrix of T observations on M jointly dependent variables, C is a $M\times M$ nonsingular matrix of coefficients associated with them, X is a $T\times\Lambda$ matrix, with full column rank, of T observations on Λ predetermined variables, B is a $\Lambda\times M$ matrix of their coefficients and U is a $T\times M$ matrix of unobservable structural disturbances with

$$\begin{aligned} &E(U)=0 \\ &\frac{1}{T}E(U'U)=\Sigma \end{aligned} \tag{2.2}$$

$\Sigma=\|\sigma_{ij}\|$ being a $M\times M$ nonsingular matrix.

We assume that the multivariate stochastic processes generating predetermined variables and structural disturbances are independent and the matrix $\frac{1}{T}X'X$ possesses a finite probability limit which is positive definite.

Further, it is assumed that the reduced form of the model (2.1) exists and there are sufficient zero-one type restrictions on the coefficient matrices C and B so that all the equations are identifiable and can be written as

$$\begin{aligned} y_i&=Y_i\gamma_i+X_i\beta_i+u_i \\ &=A_i\delta_i+u_i \qquad A_i=(Y_i\ X_i) \end{aligned} \tag{2.3}$$

$$(i=1, 2, \cdots, M) \qquad \delta_i=\begin{pmatrix}\gamma_i\\ \beta_i\end{pmatrix}$$

where $\boldsymbol{y}_i$ is a $T\times 1$ vector of T observations on the jointly dependent variable to be explained in the ith equation, $\boldsymbol{Y}_i$ and $\boldsymbol{X}_i$ are $T\times m_i$ and $T\times l_i$ matrices of T observations on $m_i(<M)$ explanatory jointly dependent and $l_i(\leq \Lambda)$ explanatory predetermined variables respectively and $\boldsymbol{u}_i$ is the ith column vector of $\boldsymbol{U}$.

Writing all the equations compactly, we have

$$\begin{pmatrix}\boldsymbol{y}_1\\ \boldsymbol{y}_2\\ \vdots\\ \boldsymbol{y}_M\end{pmatrix}=\begin{pmatrix}\boldsymbol{A}_1 & \boldsymbol{0}\cdots\boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}_2\cdots\boldsymbol{0}\\ \vdots & \vdots \quad \vdots\\ \boldsymbol{0} & \boldsymbol{0}\cdots\boldsymbol{A}_M\end{pmatrix}\begin{pmatrix}\delta_1\\ \delta_2\\ \vdots\\ \delta_M\end{pmatrix}+\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\\ \vdots\\ \boldsymbol{u}_M\end{pmatrix}$$

or

$$\boldsymbol{y}=\boldsymbol{A}\delta+\boldsymbol{u} \tag{2.4}$$

We now seek a linear and consistent estimator $\hat{\delta}$ of δ, that is,

$$\hat{\delta}=\boldsymbol{L}\boldsymbol{y} \tag{2.5}$$

with

$$\text{plim}\sqrt{T}(\hat{\delta}-\delta)=\boldsymbol{0} \tag{2.6}$$

where $\boldsymbol{L}$ is an arbitary matrix of order $N\times MT$ with $N=\sum_{i=1}^{M}(m_i+l_i)$.

The requirement (2.6) implies that

$$\text{plim}\sqrt{T}[(\boldsymbol{L}\boldsymbol{A}-\boldsymbol{I}_N)\delta+\boldsymbol{L}\boldsymbol{u}]=\boldsymbol{0} \tag{2.7}$$

for which the sufficient conditions are

$$\text{plim}\boldsymbol{L}\boldsymbol{A}=\boldsymbol{I}_N \tag{2.3}$$

$$\text{plim}\sqrt{T}\boldsymbol{L}\boldsymbol{u}=\boldsymbol{0}. \tag{2.9}$$

An obvious choice of $\boldsymbol{L}$ that satisfies (2.8) is given by

$$\boldsymbol{L}=(\boldsymbol{A}'\boldsymbol{S}_1\boldsymbol{A})^{-1}\boldsymbol{A}'\boldsymbol{S}_2 \tag{2.10}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are $MT\times MT$ matrices such that

(i) $\boldsymbol{A}'\boldsymbol{S}_1\boldsymbol{A}$ is invertible,

(ii) $\text{plim}\frac{1}{T}\boldsymbol{A}'\boldsymbol{S}_1\boldsymbol{A}=\text{plim}\frac{1}{T}\boldsymbol{A}'\boldsymbol{S}_2\boldsymbol{A}$ exist and invertible,

(iii) $\text{plim}\frac{1}{\sqrt{T}}\boldsymbol{A}'\boldsymbol{S}_2\boldsymbol{u}=\boldsymbol{0}$.

Notice that the condition (iii) implies (2.9).

Thus from (2.5) and (2.10) we find a class of linear and consistent estimators for δ and term it as 'double S-class':

$$\hat{\delta}=(A'S_1A)^{-1}A'S_2y \tag{2.11}$$

where S_1 and S_2 are the characterizing matrices satisfying condition (i), (ii) and (iii).

The variance-covariance matrix of the asymptotic distribution of $\sqrt{T}(\hat{\delta}-\delta)$ is given by

$$\begin{aligned} &\operatorname{plim} T(\hat{\delta}-\delta)(\hat{\delta}-\delta)' \\ &=\operatorname{plim} T(A'S_1A)^{-1}A'S_2uu'S_2'A(A'S_1A)^{-1} \\ &=\operatorname{plim} T(A'S_1A)^{-1}\cdot\operatorname{plim}\frac{1}{T}A'S_2(\Sigma\otimes I_T)S_2'A\cdot\operatorname{plim}T(A'S_1A)^{-1} \end{aligned} \tag{2.12}$$

provided the conditions (i), (ii) and (iii) hold.

It may be observed that while choosing S_2 sometimes the condition (iii) may be slightly difficult to check. In such cases suppose we can determine two matrices P and P^* such that both of them have MT rows and a number of columns independent of T. Now if we write $S_2=P^*P'$, then the condition (iii) is satisfied when

$$\operatorname{plim}\frac{1}{\sqrt{T}}P'u=0. \tag{2.13}$$

3. Some Special Cases

Now we demonstrate that all the known estimators are particular cases of the double S-class.

Let us first consider the full information estimators. For this purpose we denote a consistent estimator of Σ by $\hat{\Sigma}$ and write

$$S_\alpha=\begin{pmatrix} S_\alpha(1,1) & S_\alpha(1,2)\cdots S_\alpha(1,M) \\ S_\alpha(2,1) & S_\alpha(2,2)\cdots S_\alpha(2,M) \\ \vdots & \vdots \qquad\qquad \vdots \\ S_\alpha(M,1) & S_\alpha(M,2)\cdots S_\alpha(M,M) \end{pmatrix}$$

where $S_\alpha(i,j)$ is the (i,j) th submatrix, of order $T\times T$, of $S_\alpha(\alpha=1,2$ and $i,j=1,2,\cdots,M)$.

If we set

$$(3.1)\qquad \begin{aligned} S_1(i,j) &= \hat{\sigma}^{ij}[(1-k_{ij})I_T + k_{ij}X(X'X)^{-1}X'] \\ S_2(i,j) &= \hat{\sigma}^{ij}[(1-k_i^*)I_T + k_i^*X(X'X)^{-1}X'] \end{aligned}$$

in (2.11), we obtain the generalized double k-class estimator, proposed by Roy and Srivastava (1966), of which the systems k-class estimators of Savin (1973) and three stage least squares as well as iterative three stage least squares estimators of Zellner and Theil (1963) are special cases. Here k_{ij} and $k_i^*(i,j=1,2,\cdots,M)$ are characterizing scalars and $\hat{\Sigma}^{-1} = \|\hat{\sigma}_{ij}\|^{-1} = \|\hat{\sigma}^{ij}\|$

When

$$(3.2)\qquad S_1(i,j) = S_2(i,j) = \hat{\sigma}^{ij}[(1-k_{ij})I_T + k_{ij}X(X'X)^{-1}X']$$

we find **K**-matrix-class estimators, forwarded by Scharf (1976), of which full information maximum likelihood estimator of Chow (1969) and linearized full information maximum likelihood estimator of Dhrymes (1973) and Srivastava (1971), are particular cases provided disturbances follow a multivariate normal probability law.

When $T<\Lambda$, the matrix X is not of full column rank. In such a case, Swamy and Holmes (1971) suggested the generalized three stage least squares estimator which is also a member of double S-class with

$$(3.3)\qquad S_1(i,j) = S_2(i,j) = \hat{\sigma}^{ij}X(X'X)^{-1}X'$$

where $(X'X)^{-1}$ denotes the generalized inverse of $X'X$. Next, we consider the limited information methods for which we take

$$(3.4)\qquad S_1(i,j) = S_2(i,j) = 0 \text{ for } i \neq j$$

and confine attention to the estimation of δ_i, the coefficient vector in the ith equation of the model.

Using (3.4) we get from (2.11) the double S-class estimator of δ_i as

$$(3.5)\qquad \hat{\delta}_i = [A_i'S_1(i,i)A_i]^{-1}A_i'S_2(i,i)y_i.$$

Putting

$$(3.6)\qquad \begin{aligned} S_1(i,i) &= (1-k_1)I_T + k_1X(X'X)^{-1}X' \\ S_2(i,i) &= (1-k_2)I_T + k_2X(X'X)^{-1}X' \end{aligned}$$

in (3.5) we obtain the double k-class estimators, presented by Nagar (1962), interesting particular cases of which are two stage least squares, limited information maximum likelihood (assuming normality of disturbances), k-class and h-class; see Theil(1961) for the description.

When X is not of full column rank, the modified two stage least squares estimator of Amemiya(1966) becomes a member of double S-class if we substitute

$$S_1(i,i)=S_2(i,i)=Z(Z'Z)^{-1}Z' \quad (3.7)$$

where Z is a matrix of T observations on variables the number of which is equal to the rank of X.

If Z in (3.7) is a proper subset of predeterimined variables or a basis for a proper subspace of the space spanned by the predetermined variables such as a subset of the principal components of the predetermined variables, we get the truncated two stage least squares estimator; see, e.g., Fisher(1965), Mitchell and Fisher(1970) and Srivastava(1975).

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