Multi-Line System with the Switching Rules

Y. K. Ahn*

1. Introduction

The general description concerning the multi-line systems with the switching rules have been presented by Ahn. In this paper we derive some steady state probabilities for three mainly different kinds of systems.

We assume that all of the systems are to have poisson input with the finite mean $1/\lambda$ and two servers with the negative exponentional service distribution where each has its own mean $1/u_i$.

The notations follow the paper (Ahn, 1979).

2. Matrix Equation Method

Let $p_{m,n}(t)=\Pr(N_1(t)=m,\ N_2(t)=n)$

for $m=0, 1, \cdots, M_1+1$

$n=0, 1, \cdots, M_2+1$

where $M_1+M_2=M$ is the waiting room capacity.

When we assume that the reneging distribution is exponentional and the balking distribution depends on the state, then it is possible to set up a ballance equation as the following matrix equation.

$$pA=0,\ p\epsilon=1 \text{ and } 0\leq p\leq 1 \tag{2-1}$$

where $\epsilon=(1,\cdots,1)^T$ and no of 1's is $(M_1+2)(M_2+2)$ and 0 is a vector where all elements are zero.

The many classical methods used to solve equation (2-1) have involved

* Hongneung Kikye Jaejakso, Seoul, Korea.

finding the probability generating function.

It also appears that in many problems the Λ-matrix has a block-triangular structure (see, Definition 2. 1 below) after an appropriate ordering of states (Disney, 1975) In this paper mainly we discuss systems in which has a structure as follows.

The systems with the exponential ranging and state dependent balking give this this kind of structure.

Definition 2. 1.

$$\Lambda=\begin{pmatrix} \Lambda_1 & \Lambda_3 & & & & \\ \Lambda_2 & \Lambda_4 & \Lambda_3 & & & \\ & \Lambda_5 & \Lambda_4 & \Lambda_3 & & \\ & & & \cdot & & \\ & & & & \cdot & \Lambda_3 \\ & & & & \Lambda_5 & \Lambda_6 \end{pmatrix}$$

where all submatrices are non-singular with same rank. Note that two marginal probabilities $\sum_{k=0}^{M_1+1} p_{kn}$ and $\sum_{k=0}^{M_2+1} p_{nk}$ are denoted by $p._n$ and $p_n.$ respectively.

Arranging the steady state probability p according to the structure of Λ and the Equation (2-1)

We define

$$p=(p_0, \cdots, p_k)$$

and

$$p_{k+1}=p_k(\Lambda_6-\Lambda_4)\Lambda_5^{-1} \tag{2-2}$$

Lemma 2. 1. Let $X_n=(p_n, p_{n+1})$ for $n=0. 1, \cdots, k$

Then $X_{n+1}=X_0\Lambda^n{}_0$ for $n=0, \cdots\cdots\cdots\cdots\cdots, k,$

where

$$\Lambda_0=-\begin{pmatrix} 0, & \Lambda_3\Lambda_5^{-1} \\ -1, & \Lambda_4\Lambda_5^{-1} \end{pmatrix} \tag{2-3}$$

and I is the identity matrix with the rank same as that of Λ_i's.

〈**Proof**〉 $p=0$ implies that

$$p_0\Lambda_1+p_1\Lambda_2=0$$

$$p_n\Lambda_3+p_{n+1}\Lambda_4+p_{n+2}\Lambda_5=0 \text{ for } n=0, 1, \cdots, k-2 \tag{2-4}$$

$$p_{k-1}\Lambda_3+p_k\Lambda_6=0$$

Using the definition of p_{k+1}, we have

$$p_n\Lambda_3+p_{n+1}\Lambda_4+p_{n+2}\Lambda_5=0 \text{ for } n=0, 1, \cdots, k-1 \qquad (2\text{-}5)$$

Multiply Equation (2-5) by Λ_5^{-1} from the right side. Then we have

$$p_n\Lambda_3\Lambda_5^{-1}+p_{n+1}\Lambda_4\Lambda_5^{-1}+p_{n+1}=0,$$

which is the same as Equation (2-3).

Remark 2. 1. The calcuation of X_0 gives the steady state probabilities. Instead of trying to get p, we derive marginal probabilities.

Example 2. 1. System with Bivariate Poisson Input (Hunter, 1971).

The input process to the i^{th} server is the sum of two independent Poisson processes with rates λ_0 and λ_i for $i=1, 2$. This process is called the bivariate Poisson process. The application of this process to the reliability systems is well known (Barlow and Proschan, 1975, page 135).

Let $p_i=(p_{i0}, \cdots, p_{iM2+1})$ for $i=0, 1, \cdots, M_1+1$. Then Equation (2-1) has a matrix given in Definition 2. 1. The submatrices satisfy the following formula.

$$\Lambda_1=\begin{pmatrix} \lambda_0+\lambda, & -\lambda_2, & & & \\ -u_2, & \lambda+\lambda_0+u_2, & \ddots & & \\ & -u_2 & \ddots & \ddots & \\ & & \ddots & \ddots, & -\lambda_2 \\ & & & -u_2, & \lambda_0+\lambda_1+u_2 \end{pmatrix}$$

$$\Lambda_2=\Lambda_5=-u_1I.$$

$$\Lambda_3=-\begin{pmatrix} \lambda_1, & \lambda_0, & & \\ & \lambda_1, & \ddots & \\ & & \ddots & \ddots, \; \lambda_0 \\ & & & \lambda_0+\lambda_1 \end{pmatrix}$$

$$
\Lambda_4=\begin{pmatrix} \lambda+\lambda^0+u_1, & -\lambda_2, & & & \\ -u_2, & \lambda+\lambda_1+u, & -\lambda_2, & & \\ & -u_2, & \ddots & & \\ & & \ddots & ; & \lambda_2 \\ & & & -u_2, & \lambda_0+\lambda_1+n \end{pmatrix}
$$

$$
\Lambda_6=\begin{pmatrix} \lambda+\lambda_2+u_1, & -(\lambda+\lambda_2), & & & \\ -u_2, & \lambda+\lambda_2+u, & -(\lambda+\lambda_2), & & \\ & -u_2, & \ddots & & \\ & & \ddots & , & -(\lambda+\lambda_2) \\ & & , & -u_2, & u \end{pmatrix}
$$

where $\lambda=\lambda_1+\lambda_2$ and Λ_i's are $(M_1+1)\times(M_1+1)$ matrices.

Now we define $\epsilon=(1, \cdots, 1)^T$ so that the multiplication of matrices is possible by having some number of 1's.

Hence,

$$\Lambda_3\epsilon=\Lambda_1\epsilon=(\lambda_0+\lambda_1)\epsilon$$

$$\Lambda_0\epsilon=\Lambda_2\epsilon=\Lambda_5\epsilon=-u_1\epsilon$$

$$\Lambda_4\epsilon=(\lambda_0+\lambda_1+u_1)\epsilon$$

Using the above equations, Equation (2-4) becomes

$$
\begin{aligned}
&(\lambda_0+\lambda_1)\, p_0\epsilon=u_1\, p_1\epsilon \\
&(\lambda_0+\lambda_1+u_1)\, p_{n+1}\epsilon=(\lambda_0+\lambda_1)\, p_n\epsilon+u_1\, p_{n+2}\epsilon \quad \text{for } n=0, 1, \cdots, M_1-1 \\
&(\lambda_0+\lambda_1)\, p_{M_1}\epsilon=u_1\, p_{M_1+1} \qquad (2\text{-}6)
\end{aligned}
$$

Since $p_n\epsilon=\sum_{k=0}^{M_2+1} p_{nk},\ p_n\epsilon=p_{n\cdot}$.

Therefore, Equation (2-6) which is the same as the balance equation of a birth-death process, yields the marginal probabilities as follows: If we

$$p_{n\cdot}=p_{0\cdot}\ \frac{(\lambda_0+\lambda_1)^n}{u_1} \quad \text{for } n=0, 1, \cdots, M_1+1$$

rearrange the order of p_{ij}'s, then we have

$$p_{\cdot n}=p_{\cdot 0}\frac{(\lambda_0+\lambda_2)^n}{u_2} \quad \text{for } n=0, 1, \cdots M_2+1$$

Remark 2. 2. The marginal probabilities given in Example 2. 1 are intuitively clear, since two servers act independently and the input process into each server is a Poisson process.

In particular, consider a matrix whose submatrices satisfy the following formula

$$A_2=A_5=-uI, \ A_3=-I.$$

Then A_0 given in Lemma 2. 1 reduces to

$$A_0=-\begin{pmatrix} 0, & \frac{\lambda}{u_1}I \\ -I, & \frac{1}{u_1}A_4 \end{pmatrix}$$

Using the boundary conditions, i. e., balance equations containing p_0, p_1, and p_k, p_{k+1}, it is possible to get p up to a constant multiple which can be calculated by the normalizing condition.

As before the basic difficulty arises from the problem of finding a power of A_0.

3. **Overflow System** (See Ahn)

The balance equations become

$$p_{00}=u_1 p_{10}+u_2 p_{01}$$

$$(\lambda+u_1)\, p_{n0}=u_1 p_{n+1,0}+\lambda p_{n-1,0}+u_2 p_{n,1}, \quad \text{for } 0 \ n \ M_1+1$$

$$(\lambda+u_1)\, p_{u_1+1,0}=u_2 p_{M_1+1,1}+\lambda p_{M_1,0}$$

$$(\lambda+u_2)\, p_{0n}=u_1 p_{1,n}+u_2 p_{0,n+1} \quad \text{for } 0<n<M_2+1$$

$$(\lambda+u)\, p_{m,n}=u_1 p_{m+1,n}+u_2 p_{m,n+1}+\lambda p_{m-1,n} \quad \text{for } 0<n<M_2+1,\ 0<m<M_1+1$$

$$(\lambda+u)+M_1+1, n=\lambda p_{M_1,n}+u_2 p_{M_1+1,n+1}+\lambda p_{M_1+1,n-1} \quad \text{for } 0<n<M_2+1$$

$$u_1 p_{1,M_2+1}=(\lambda+u_2)\, p_{0,M_2+1}$$

$$(\lambda+u)\, p_{m,M_2+1}=u_1 p_{m+1,M_2+1}+\lambda p_{m-1,M_2+1} \quad \text{for } 0<m<M_1+1$$

$u p_{M_1+1, M_2+1} = \lambda p_{M_1, M_2+1} + \lambda p_{M_1+1, M_2}$

Let $p_n = (p_{n,0}, \cdots, p_{n, M_2+1})$ for $n = 0, 1, \cdots, M_1+1$

Lemma 3.1

$$p_{n\cdot} = (1-\rho_1)\rho_1{}^n / (1-(\rho_1)^{M_1+2}) \text{ for } n = 0, 1, \cdots, M_1+1 \qquad (3\text{-}1)$$

and

$$p_{\cdot n} = \rho_2 p_{M_1+1, n-1} \text{ for } n = 1, \cdots, M_2+1 \qquad (3\text{-}2)$$

⟨**Proof**⟩ The balance equations can be written as the matrix from. The submatrices are;

$$\Lambda_1 = \begin{pmatrix} \lambda & & & & \\ -u_2, & \lambda+u_2, & & & \\ & -u_2, & \lambda+u_2, & & \\ & & \ddots & \ddots & \\ & & , & -u_2, & \lambda+u_2 \end{pmatrix}$$

$$\Lambda_3 = \begin{pmatrix} \lambda+u_1, & & & \\ -u_2, & \lambda+u_,, & & \\ & -u_2, & \ddots & \\ & , & -u_2, & \lambda+u \end{pmatrix}$$

$$\Lambda_6 = \begin{pmatrix} \lambda+u_1, & -\lambda, & & & \\ -u_2, & \lambda+u, & -\lambda, & & \\ , & -u_2, & \ddots & \ddots & \\ & & , & , & -\lambda \\ & & & -u_2, & u \end{pmatrix}$$

$\Lambda_2 = \Lambda_5 = -u_1 I.$

$\Lambda_3 = -\lambda I.$

Equation (3-1) can be derived as Example 2. 1.

Now define the vectors $p_n' = (p_{0n}, \cdots, p_{M_1+1, n})$, for $n = 0, 1, \cdots, M_2+1$ Then the vector $p' = (p_0', \cdots, p'_{M_2+1})$ satisfy the matrix equation $p'\Lambda = 0$ where matrix is the form given in the Definition 2. 1.

The submatrices have the following forms:

$$\Lambda_1 = \begin{pmatrix} \lambda, & -\lambda, & & & \\ -u_1, & \lambda+u_1, & -\lambda, & & \\ & -u_1, & \ddots & \ddots & \\ & & \ddots & \ddots & -\lambda \\ & & & -u_1, & \lambda+u_1 \end{pmatrix}$$

$\Lambda_2 = \Lambda_5 = -u_2 I.$

$$\Lambda_3 = \begin{pmatrix} 0 & & & \\ & \ddots & & 0, \\ & & \ddots & \\ & & & -\lambda \end{pmatrix}$$

$$\Lambda_4 = \begin{pmatrix} \lambda+u_2, & -\lambda, & & & \\ -u_1, & \lambda+u, & -\lambda, & & \\ & & \ddots & \ddots & \\ & & & \ddots & -\lambda \\ & & & -u_1, & \lambda+u \end{pmatrix}$$

$$\Lambda_6 = \begin{pmatrix} \lambda+u_2, & -\lambda, & & & \\ -u_1, & \lambda+u, & & & \\ & -u_1, & \ddots & & \\ & & \ddots & \ddots & -\lambda \\ & & & -u_1, & u \end{pmatrix}$$

Then

$$\Lambda_1 \epsilon = \lambda \epsilon_1$$
$$\Lambda_3 \epsilon = -\lambda \epsilon_1$$
$$\Lambda_4 \epsilon = u_2 \epsilon + \lambda \epsilon_1$$
$$\Lambda_6 \epsilon = u_2 \epsilon, \text{ where } 1 = (0, , 0, \cdots 1)^T$$

Multiply Equation (2-4) as in Example, 2. 1, then we have

$$\lambda p_{M_1+1, 0} = u_2 p_{\cdot 1}$$
$$\lambda p_{M_1+1, n} = u_2 p_{\cdot n+1} + \lambda p_{M_1+1, n+1} - u_2 p_{\cdot n+2} \text{ for } n = 0, 1, \cdots, M_2 - 1$$
$$\lambda p_{M_1+1, M_2} = u_2 p_{\cdot M_2+1}$$

The inductive argument proves Equation (3-2).

Remark 3. 1. Ghirtis (1968) said that Equation (3-2) can be derived by using the probability generating function, which is usually a hard method. 2. 2. Furthermore, the normalizing condition gives,

$$p_{\cdot 0} = 1 - \rho_2 (p_{M_1+1\cdot} - p_{M_1+1, M_2+1}) \tag{3-3}$$

Now we derive p_{m, M_2+1} in terms of p_{0, M_2+1}. Considering the balance equations having p_{m, M_2+1}, we have the following lemma.

Lemma 3. 2. Let w_1 and w_2 be the two roots of the following equation; $u_1 w^2 - (\lambda + u) w + \lambda = 0$. Then

$$p_{m, M_2+1} = p_{0, M_2+1} e_m \text{ for } m = 1, \cdots, M_1 + 1, \tag{3-4}$$

where $e_m = v_m - v_{m-1}$

and $v_m = \dfrac{{w_1}^{m+1} - {w_2}^{m+1}}{w_1 - w_2}$

〈**Proof**〉 When $m = 1$,

$$v_1 = (w_1 + w_2) = \frac{1}{u_1}(\lambda + u) \text{ and } e_1 = \frac{1}{u_1}(\lambda + u_2).$$

Therefore $p_{1, M_2+1} = p_{0, M_2+1} e_1$ satisfies the balance equation.

When $m = 2$,

$$v_2 = (w_1 + w_2)^2 - w_1 w_2 = \frac{1}{u_1}\left(\frac{(\lambda+u)^2}{u_1} - \lambda\right)$$

$$e_2 = \frac{1}{u_1}\left(\frac{(\lambda+u)(\lambda+u_2)}{u_1} - \lambda\right), \text{ so } p_{2, M_2+1} \text{ satisfies.}$$

Assume that Equation (3-4) is true for m. Then for $m+1$ it is sufficient to show that

$$u_1 e_{k+1} = (\lambda+u) e_k - \lambda e_{k-1} \tag{3-5}$$

Now consider the two roots, then

$$u_1(w_1^{m+2} - w_2^{m+2}) = w_1^m\{(\lambda+u)w_1 - \lambda\} - w_2^m\{(\lambda+u)w_2 - \lambda\}$$
$$= (\lambda+u)w_1^{m+1} - \lambda w_1^m - (\lambda+u)w_2^{m+}{}_1 + \lambda w_2^m \tag{3-6}$$

Since Fquation (3-6) implies Equation (3-5), using the mathematical induction we show that Equation (3-4) satisfies equations

$$u_1 p_{1,\ M_2+1} = (\lambda+u_2)\, p_{0,\ M_2+1}$$

and

$$(\lambda+u)\, p_{m,\ M_2+1} = u_1 p_{m+1,\ M_2+1} + \lambda p_{m-1,\ M_2+1} \text{ for } 0 < m < M_1+1$$

The uniqueness of solutions for the above equations yields the proof.

Remark 3. 2. Ghirtis(1968) solved p recursively after getting the Lemma 3. 2. However, we derive the probabilities $p_{\cdot n}$ by a different method which will be di:cussed later.

3. 1 Subnetwork Method

Clark and Disney (1967) have shown that the input process into the second server is a renewal process. Therefore, it is possible to derive $p_{\cdot n}$ separately by using this input.

Definition 3. 1. Define T_e to be the time between two consecutive arrivals into the second server.

First we need the L.S.T. of T_e

Theorem 3. 1.

$$A_e{}^*(u_2) = 1 + \frac{1}{\rho_2}\left(\frac{v_{M_1+1}}{e_{M_1+1}}\right)^{-1}, \text{ where} \tag{3-7}$$

$A_e{}^*(\theta)$ is the L.S.T. of T_e.

Let $M_e(t)$ be the renewal function of T_e, then

$$\lim_{t\to\infty} \frac{M_e(t+\Delta) - M_e(t)}{\Delta} = \frac{1}{A_e{}^{(1)}(0)} = p_{M_1+1} \tag{3-8}$$

where the superscripts of function mean the number of the differentiation with respect to θ.

Conditioning on the points of arrivals into the second server, define the steady state probabilities

$$\Pi_k^{(2)}=\Pr(N_2=k\,|\,\text{there is an arrival})\ \text{ for } k=0,1,\cdots,M_2+1$$

Solving the equations set up by the supplementary variable method (Ahn) we have,

$$\frac{\Pi_{M_2}^{(2)}}{\Pi_{M_2+1}^{(2)}}=\frac{1-A_e^*(u_2)}{A_e^*(u_2)} \tag{3-9}$$

Using Equation (3-8), Theorem 4. 4. (Ahn) yields

$$\Pi_n^{(2)}=\frac{u_2\,p_{\cdot n+1}}{\lambda p_{M+1\cdot}}\quad \text{for } n=0,1,\cdots,M_2 \tag{3-10}$$

and

$$\Pi_{2+1}^{(2)}=1-\frac{u_2}{\lambda p_{M+1\cdot}}\sum_{n=1}^{M_2+1}p_{\cdot n}=\frac{p_{M_1+1,\,M_2+1}}{p_{M_1+1}} \tag{3-11}$$

Since $\quad p_{\cdot M_2+1}=p_{0,\,M_2+1}\sum_{n=0}^{M_1+1}e_n=p_{0,\,M_2+1}v_{M_1+1}$, we have

$$\Pi_{M_2}^{(2)}=\frac{1}{\rho_2}\ \frac{p_{0,\,M_2+1}}{p_{m+1}}v_{M_1+1} \tag{3-12}$$

Therefore, Equations (3-9), (3-11) and (3-12) give the proof.

Corollary 3. 1.

$$p_{\cdot n+1}=D(M_2-n)-D(M_2-1-n)\frac{\rho_2\,p_{M_1+1,\,M_2+1}}{A_e^*(u_2)} \tag{3-13}$$

for $n=0,1,\cdots,M_2-1$

$$p_{\cdot M_2+1}=(1-A_e^*(u_2))\frac{\rho_2\,p_{M_1+1,\,M_2+1}}{A_e^*(u_2)} \tag{3-14}$$

$$p_{M_1+1,\,M2+1}=\frac{A_e^*(u_2)\,p_{M_1+1}}{D(M_2)} \tag{3-15}$$

where $\quad D(o)=1$

$$D(t)=\sum_{j=1}^{k}\ \sum_{i,\,+\cdots i_j=k}d_i\cdots\cdots d_{ij}\ \text{ for } k=1,2,\cdots$$

$$d_1=\frac{1+A^{(1)}(u)}{A^*(u)}$$

$$d_k=\frac{(-u)^kA^{(k)}(u)}{k!\,A^*(u)}$$

〈**Proof**〉 Inductive arguments on the equations (Section 4, Ahn) give

$$\Pi_n^{(2)}=\{D(M_2-n)-D(M_2-n-1)\}\frac{\Pi^{(2)}M_2+1}{A_e^*(u_2)} \text{ for } n=0, 1, \cdots, M_2-2$$

Then Equations (3-10) and (3-11) yield Equations (3-13) and (3-14).

Summing all $p._n$'s, we have Equation (3-15) by using normalizing conditions.

Remark. 3. 3. It is very complicated to get $A_e^*(\theta)$ using the methods of Cinlar and Disney (1967); however, we need $A_e^*(\theta)$ only when $\theta=u_2$. Furthermore, $\Pi_n^{(2)}$ is not the same as p_n, since $\Pi_n^{(2)}$'s are the conditional probabilities which are different from $\Pi._n$'s.

3. 2. General Overflow System

We generalize the overflow system by adding two variables; (i) T_t is the r.v. of traveling time from the first queue into the second queue when $N_1=M_1+1$, (ii) whether joining the second queue or not depends on b.

Define the L.S.T. of T_t by $A_t^*(\theta)$ with the finite mean $1/\lambda_1$.

Then the L.S.T. of T_e+T_e becomes $A_e^*(\theta)A_t^*(\theta)$ by assuming independence.

Let $p_{m,n}(u,t)du=\Pr(N_1(t)=m, L_2(t)=n,\ u\ \ U(t)\ \ u+du)$.

Then we have for $b=\epsilon$,

$$(\theta-\lambda)\,p^*_{00}(\theta)=p_{00}(0)-u_1p^*_{10}(\theta)-u_2p^*_{01}(\theta)$$

$$(\theta-\lambda-u_2)\,p^*_{0n}(\theta)=p_{0n}(0)-u_1p^*_{1,n}(\theta)-u_2p^*_{0,n+1}(\theta)$$
$$-A_\tau^*(\theta)\,p_{0,n-1}(0) \quad \text{for } 0<n<M_2+1$$

where $A_\tau^*(\theta)=A_e^*(\theta)A_t^*(\theta)$

$$(\theta-\lambda-u_2)\,p^*_{0,M_2+1}(\theta)=p_{0,M_2+1}(0)-1\,p^*_{0,M_2+1}(\theta)$$
$$-A_\tau^*(\theta)\ \ p_{0,M_2}(0)+p_{0,M_2+1}(0)$$

$$(\theta-\lambda u_1)\,p^*_{m,0}(\theta)=p_{m,0}(0)-u_1p^*_{m+1,0}(\theta)-u_2p^*_{m,1}(\theta)-\lambda p^*_{m-1,0}(\theta)$$
$$\text{for } 0<m<M_1+1$$

$$(\theta-u_1)\,p^*_{M_1+1,0}(\theta)=p_{M_1+1,0}(0)-u_2p^*_{M_1+1,1}(\theta)-\lambda p^*_{M_1,0}(\theta)$$

$$(\theta-\lambda-u)\,p^*_{m,n}(\theta)=p_{m,n}(0)-u_1p^*_{m+1,n}(\theta)-u_2p^*_{m,n+1}(\theta)$$
$$-\ p^*_{m-1,n}(\theta)-A_\tau^*(\theta)\,p_{m,n-1}(0) \quad \text{for } 0<m<M_1+1$$

and $0<n<M_2+1$

$$(\theta-\lambda-u)\,p^*_{m,M_2+1}(\theta)=p_{m,M_2+1}(0)-u_2p^*_{m_2+1,M+1}(\theta)-\lambda p^*_{m-1,N}(\theta)$$
$$-A_\tau^*(\theta)\{p_{m,M_2}(0)+p_{m,M_2+1}(0)\} \quad \text{for } 0<m<M_1+1$$

$$(\theta-u)\, p^*_{M_1+1,n}(\theta)=p_{M_1+1,n}((0)-u_2 p^*_{M_1+1,n+1}(\theta)-A_\tau^*(\theta)\, p_{M_1+1,n-1}(0)$$

for $0<m<M_2+1$

$$(\theta-u)\, p^*_{M_1+1,M_2+1}(\theta)=p_{M_1+1,M_2+1}(0)-A_\tau^*(\theta)\{p_{M_1+1,M_2}(0)+p_{M_1+1,M_2+1}(0)\}$$

Theorem 3. 2. $p_{n\cdot}=p_{0\cdot}\rho_1^n$ for $n=0, 1, \cdots, M_1+1.$ (3-16)

Furthermore, $p_{\cdot n}$ satisfies equations given in Corollary 3. 1. by replacing $A_e^*(u_2)$ by $A_\tau^*(u_2)$.

〈**Proof**〉 Adding the corresponding terms, we have $\lambda p_{0\cdot}=u_1 p_1$

$$(\lambda+u_1)\, p_{m\cdot}=u_1 p_{m+1\cdot}+\lambda p_{m-1\cdot} \text{ for } 0<m<M_2+1$$

Hence, Equation (3-16) follows. For $p_{\cdot n}$'s we have

$$\theta p_{\cdot 0}^*(\theta)=p_{\cdot 0}(0)-u_2 p_{\cdot 1}^*(\theta)$$

$$(\theta-u_2)\, p_{\cdot n}^*(\theta)=p_{\cdot n}(0)-u_2 p_{\cdot n+1}^*(\theta)-A_\tau^*(\theta)\, p_{\cdot n-1}(0) \qquad (3\text{-}17)$$

for $0<n<u_2+1$

$$(\theta-u_2)\, p_{\cdot M_2+1}^*(\theta)=p_{\cdot M_2+1}(0)-A_\tau^*(\theta)\{p_{\cdot M_2+1}(0)+p_{\cdot M_2}(0)\}.$$

The above equations are the same as those given in the proof of Theorem 4. 1 (Ahn) by assuming $b=\epsilon$ and replacing $A^*(\theta)$ by $A_\tau^*(\theta)$. Therefore, the proof follows.

Remark 3. 4. Since the decomposition of the system into two subsystems as Section 3. 1 is possible, Theorem 3. 2 is intuitively clear. However, it is impossible to obtain A matrix, thus we need the supplementary variable method used above.

4. Instantaneous Jockeying System

Consider the M.S.S.R. with the following structure

(a) $r(n)=1$ if $m_1<m_2$

1 if $m_1=m_2$

0 if $m_1>m_2$

where $n=(m_1,m_2)$

(b) $b(n)=0$ if $n=(M_1+1, M_1+1)$

1 otherwise

(c) While waiting in the i^{th} queue, if the number of items of another queue is less than that of the i^{th} queue by 2, then an item in the i^{th} queue moves into another queue for $i=1, 2$.

Remark 4. 1. The possible states are (x,x), $(x+1,x)$, $(x,x+1)$ and (M_1+1, M_1+1), for $x=0, 1, \cdots, M_1$. When $r_1=1$ and $M_1=\infty$, then this system becomes the system given by Haight (1958).

The balance equations become

$$\lambda p_{00}=u_1 p_{10}+u_2 p_{01}$$

$$(\lambda+u_1) p_{10}=u_2 p_{11}+\lambda r_1 p_{00}$$

$$(\lambda+u_2) p_{01}=u_1 p_{11}+\lambda r_2 p_{00}$$

$$(\lambda+u) p_{i,i-1}+p_{i-1,i})+u(p_{i,i+1}+p_{i+1,i}) \text{ for } 0<i<M_1+1$$

$$(\lambda+u) p_{i,i+1}=\lambda r_2 p_{ii}+u_2 p_{i+1,i+1} \text{ for } 0<i<M_1+1$$

$$(\lambda+u) p_{i+1,i}=\lambda r_1 p_{ii}+u_2 p_{i+1,i+1} \text{ for } 0<i<M_1+1$$

$$u p_{M_1+1,M_1+1}=\lambda\{p_{M_1,M_1}+p_{M_1+1,M_1}\}$$

Theorem 3. 3. $p_{k+1,k+1}=\rho^{2k} p_{11}$ (4-1)

$$p_{k,k+1}=\left(\frac{\lambda r_2}{\rho^2}+u_1\right)\frac{\rho^{2k}}{(\lambda+u)} p_{11} \tag{4-2}$$

$$p_{k+1,k}=\left(\frac{\lambda r_1}{\rho^2}+u_2\right)\frac{\rho^{2k}}{(\lambda+u)} p_{11} \text{ for } k=0, 1, \cdots, M_1$$

〈**Proof**〉 Let $p=(p_0, \cdots, p_{M_1+1})$,

where

$p_k=(p_{k,k}, p_{k+1,k}, p_{k,k+1})$, p_{M_1+2,M_1+1} and p_{M_1+1,M_1+2}

are arbitrary.

Then p satisfies Equation (2-1) where A matrix has the structure given in the Definition 2. 1.

The corresponding matrices are

$$A_1=\begin{pmatrix} -\lambda, & \lambda r_1, & \lambda r_2 \\ u_1, & -(\lambda+u_1), & 0 \\ u_2, & 0, & -(\lambda+u_2) \end{pmatrix}$$

$$A_2=A_5=\begin{pmatrix} 0 & u_2 & u_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Lambda_3=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \lambda & 0 & 0 \end{pmatrix}$$

$$\Lambda_4=\begin{pmatrix} -(\lambda+u), & \lambda r_1, & \lambda r_2 \\ u, & -(\lambda+u), & 0 \\ u, & 0, & -(\lambda+u) \end{pmatrix}$$

Multiplying Equation (2-4) by ϵ, we have

$$\lambda\{p_{k,k+1}+p_{k+1,k}\}-u\,p_{k+1,k+1}=u\{p_{k+1,k+2}+p_{k+2,k+1}\}-u\,p_{k+2;k+2} \quad \text{for } k=0,1,\cdots,M_1-1 \qquad (4\text{-}3)$$

Then

$$p_{00}=l_0\,p_{11},\ p_{10}=l_1\,p_{11} \text{ and } p_{01}=l_2\,p_{11},$$

where l_i's are defined as follow:

$$l_0=\frac{(2\lambda+u)u_1u_2}{\lambda^2(\lambda+u_1r_2+u_2r_1)}$$

$$l_1=\frac{u_2(\lambda+ur_1)}{\lambda(\lambda+u_1r_2+u_2r_1)}$$

$$l_2=\frac{u_1(\lambda+ur_2)}{\lambda(\lambda+u_1r_2+u_2r_1)}$$

Therefore, by using the last balance equations the proof follows.

Remark 4.2. The normalizing condition gives

$$p_{11}=l_0+\frac{2-\rho-\rho^{2(M_1+1)}}{\rho(1-\rho)}$$

If we consider the states at the total numbe of the items in system, then this r.v. is the same as the r.v. defined in the S.S.S.R. $(2,r,b,\alpha)$ where r is the same, $\alpha_i=0$ and

$$b_k=\begin{cases}1 & \text{if } k\leq 2M_1+1\\ 0 & \text{if } k>2M_1+1\end{cases}$$

Therefore,

$$E(N_w)=\frac{\rho}{(1-\rho)^2}\{1-2M_1+3)\rho^{2M_1+2}+2(M_1+2)\rho^{2M_1+3}\}$$

5. Self-optimizing System

Consider the system with the same r and b as the overflow system. In addition we have the jockeying rule depending on the following quantities (Yeichi, 1972).

Definition 5. 1. Define the following quantities,

R_i; The reward obtained on the successful completion of the service by i^{th} server, for $i=1, 2$.

x_i; The service charge paid for the service by i^{th} server, for $i=1, 2$.

x; The rate of waiting time (including the time in service) cost per unit time for each item which joins the system.

Assume that $r^{(1)}=R_1-x_1 \geq r^{(2)}=R_2-x_2 \geq 0$, then

$$R_m^{(i)}=\left\{r^{(i)}-\left(\frac{m+1}{u_i}\right)x\right\} \text{ for } m=0, 1, \cdots, M_1 \text{ if } i=1 \text{ and}$$

$m=0, 1, \cdots, M_2$ if $i=2$, where $R_m^{(i)}$ is the expected net gain for the item at the m^{th} position in the i^{th} queue.

Definition 5. 2. If an item waiting in the second queue can get at least the same expected net gain, it will move into the first queue. If an item waiting in the first queue can get a better expected not gain, it will move into the second queue. If the above jockeying happens instantaneously, we define the above jockeying as self-optimizing jockeying rule. The system with this jockeying rule is called the self-optimizing system.

Definition 5. 3. Define $E_n=[r^{(1)}-r^{(2)}+(n+1)\frac{u_1}{u_2}]$ for $n=0, 1, \cdots, M_2+1$, where [ ] is integer function, i.e., $[y]$ is the largest integer which doesn't exceed y.

Since we have the instantaneous jockeying, the possible states are (if M_1 and M_2 are infinite):

$\{(m, 0) : m \leq E_0\}$

$\{(m, 1) : m \leq E_1\}$

$\{(m, n) : E_{n-1} \leq m \leq E_n\}$ for $m, n=2, \cdots$.

The size of the waiting rooms will give three different cases as follows:

(a) $M_1+1 \leq E_0$

(b) $M_1+1 > E_{M_2+1}$

(c) $E_0 < M_1+1 \leq E_{M_2+1}$

Remark 5. 1. The structure of the state probabilities are unaffected by a change in the value of r and also by which item changes queue due to the instantaneous jockeying. However, these factors do affect the waiting times.

5. 1 $M_1+1 \leq E_0$

If $M_1+1 \leq E_0$, then $R^{(1)}{}_{M_1} \geq R_0{}^{(R)}$. Hence, an item in the second queue makes a jockeying whenever it is possible. Therefore, the possible states are

$$\{(m,n) : m=0, 1, \cdots, M_1+1 \text{ and } n=0, 1\}$$

$$\{(M_1+1, n) : n=2, 3, \cdots, M_2+1\}$$

The balance equations become

$$\lambda p_{00}=u_1 p_{10}+u_2 p_{01}$$

$$(\lambda+u_1)\, p_{i,0}=u_1 p_{i+1,0}+u_2 p_{i,1}+\lambda p_{i-1,0} \text{ for } 0\ i\ M_1+1$$

$$(\lambda+u_2)\, p_{01}=u_1 p_{11}$$

$$(\lambda+u)\, p_{i,1}=u_1 p_{i+1,1}+\lambda p_{i-1,1} \text{ for } 0<i<M_1+1$$

$$(\lambda+u_1)\, p_{M+1,0}=u_2 p_{M_1+1,1}+\lambda p_{M_1,0}$$

$$(\lambda+u)\, p_{M_1+1,1}=u\, p_{M_1+1,2}+\lambda\{p_{M_1+1,0}+p_{M_1,1}\}$$

$$(\lambda+u)\, p_{M_1+1,n}=u\, p_{M_1+1,n}+\lambda p_{M_1+1,n-1} \text{ for } 1<n<M_2+1$$

$$u\, p_{M_1+1,M_2+1}=\lambda p_{M_1+1,M_2}$$

Theorem 5. 1. $p_{M_1+1,n}=\rho^{n-1} p_{M_1+1,1}$ for $n=1, 2, \cdots, M_2+1$ (5-1)

$$p_{m,1}=p_{M_1+1,1}\left(\frac{e_m}{e_{M_1+1}}\right) \tag{5-2}$$

$$p_{m,0}=p_{M_1+1,1}\left\{\left(1+\rho_2\frac{v_{M_1+1}}{e_{M_1+1}}\right)\rho_1{}^{M_1+1-m}-\frac{e_m}{e_{M_1+1}}\right\}$$

for $m=0, 1, \cdots, M_1+1$. (5-3)

⟨**Proof**⟩ The last two equations are the same as those of a birth-death process, therefore Equation (5-1) follows. Using Eduation (5-1), we have

$$u p_{M_1+1,1}=\lambda\{p_{M_1+1,0}+p_{M_1,1}\} \tag{5-4}$$

Let $p_k=(p_{k0}, p_{k1})$ for $k=0, 1, \cdots, M_1+1$.

Then Λ matrix has the structure defined as the Definition 2. 1. with the following submatrices:

$$\Lambda_1=\begin{bmatrix} \lambda, & 0 \\ -u_2, & \lambda+u_2 \end{bmatrix}$$

$$\Lambda_2=\Lambda_5=-u_1I,\ \Lambda_3=-\lambda I$$

$$\Lambda_4=\begin{bmatrix} \lambda+u_1, & 0 \\ -u_2, & \lambda+u \end{bmatrix}$$

$$\Lambda_5=\begin{bmatrix} \lambda+u_1, & -\lambda \\ -u_2, & u \end{bmatrix}$$

Therefore, Λ matrix is the same as the one we had for the overflow system $M_2=0$. Using Corollary 3. 1, the proof follows.

Remark 5. 2 If we take $M_2=0$, then the steady state probabilities are those of Krishnamoorthi system (Ahn). Since $p_{M_1+1,1}$ can be derived from the normalizing condition, Theorem 5. 1 gives the complete structure of the steady state probabilities.

5. 2. $M_1+1>E_{M_2+1}$

Let N_t be the r.v. of the total number of items and

$$p_n{}^t=\Pr(N_t=n) \text{ for } n=0, 1, \cdots, M_1+M_2+2.$$

Choose any integer k such that $E_0+1\leq k<M_2+1+E_{M_2}$, then there is a unique positive integer n_k which satisfies

$$E_{n_k-1}\leq k-N_k\leq E_{n_k}$$

Since the possible states for N are

$$\{(\mathrm{m},0) : m\leq E_0\}$$
$$\{(m,1) : m\leq E_1\}$$
$$\{(\mathrm{m,n}) : E_{n-1}\leq m\leq E_n, \text{ for } n=2, \cdots, M_2\}$$
$$\{(m,M_2+1) : m\geq E_{M_2}\},$$

we have

$$p_k{}^t= p_{k-n_k,n_k} \text{ if } E_0+1\leq k<M_2+1+E_{M_2} \qquad (5\text{-}5)$$
$$p_{k-M_2-,M_2+1} \text{ if } k\geq M_2+1+E_{M_2}$$

Therefore, as in Section 5. 1., we have

$$p_k{}^t = p_{E_0,1}\rho^{k-(E_0+1)} \text{ for } k=E_0+1, \cdots, M_1+M_2+2.$$

$$p_{k,1} = p_{E_0,1}\left(\frac{e_k}{e_{E_0}}\right)$$

$$p_{k,0} = p_{E_0,1}\left(1+\rho_2\frac{v_{E_0}}{e_{E_0}}\right)\rho_1{}^{E_0-k} - \frac{e_k}{e_{E_0}} \text{ for } k=0, 1, \cdots, E_0.$$

5.3 $E_0 < M_1+1 \leq E_{M_2+1}$

We can choose a positive number k_0 such that

$$E_{k_0-1} < M_1+1 \leq E_{k_0}$$

The differences in this case from Section 5.2 are in Equation (5-5) and in the possible states. The possible states are

$$\{(m, 0) : m \leq E_0\}$$
$$\{m, 1) : m \leq E_1\}$$
$$\{(m,n) : E_{n-1} \leq m \leq E_n, \text{ for } n=2, \cdots, k_0\}$$
$$\{(M_1+1, n) : n \geq k_0+1\}$$

Since we have a unique n_k such that

$$E_{n_k-1} \leq k-n_k \leq E_{n_k} \text{ for } k=E_0+1, \cdots, M_1+1+k_0,$$

we have

$$p_k{}^t = p_{k-n_k, n_k} \text{ if } k=E_0+1, \cdots, M_1+1+k_0$$
$$p_{M_1+1, k-(M_1+1)} \text{ if } k \; M_1+1+k_0+1$$

Remark 5.3. Except for the two sets of the states, $\{(m, 0) : m \leq E_0\}$ and $\{(m, 1) : m \leq E_1\}$, the total number of items uniquely determine the state of the system. This is intuitively clear, since the self-optimizing jockeying rule moves any item in the queue instantaneously.

Furthermore, if we allow an item in service to move, then we can drop the set $\{(m, 1) : m \leq E_1\}$ from the above.

REFERENCES

[1] Ahn, Y. "Multi-Line System with the Switching Rules," 1979, To be published.

[2] Barlow, R and Proschan, F. *Statistical Theory of Reliability and Life Testing*

Prcbability Models, Holt, Rinehart and Winston, New York, 1975

[3] Cinlar, E. and Disney, R. "Streams of Overflows From a Finite Queue," *Operations. Research.,* vol. 15, 1967, 131-134.

[4] Disney, R. "A Matrix Solution for the Two Server Queues with Overflow," *Managerial Science,* vol. 19, no. 3, 1972, 254-265.

[5] Disney, R. "Random flow in queuing networks: A Review and Critique," *A.I.I.E. Transactions,* Vol. 7, no. 3 1975, 268-288.

[6] Ghirtis, G. "A System of Two Servers with Limited Waiting Rooms and Certain Order of Visits," *Biometrika,* vol. 55, 1968, 223-228.

[7] Haight, F. "Two Queues in Parallel", *Biometrika,* vol. 45, 1958, 401-410.

[8] Hunter, J. "Further Studies on Two Queues in Parallel", *Austrian Journal of Statistics,* vol. 13, 1971, 83-93.

[9] Yechiali, U., "Customer's Optimal Joining Rules for GI M Squeue", *Managerial Science,* vol. 18, no. 17, 1972, 434-443.