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Some Properties of Heterogeneous Multi-server Systems with the Switching Rules

Yunkee Ahn*

Abstract:

The Classical multi-server heterogeneous queuing system can be more generalized by using the concept of the switching rules. The descriptions of these systems, the relations among the state probabilities at the various points of interest, and comparisons with the single-server system will be presented. Instead of using the imbedded markov chain we set up the simultaneous equations for the state probabilites by the supplementary variable method.

Introduction

In the past most of the research has been on multi-server systems which consist of each server performs identically. However, it is more natural to assume that each server performs differently. This system will be called the *heterogeneous multi-server system*.

Generally, the structure of queuing systems depends on the input process, the service mechanism and the behabior of items before they receive service or depart the system. Collectively, this behavior is called the *switching rules.*

In this paper some properties of the heterogeneeus server systems will be discussed. The description of systems, the relations among the state probabilities at the various points of interest, and comparisons with the single-server system will be discussed. Instead of using the imbedded markov chain we set up the simultaneous equations for the state probabilities by the supplementary variable method.

We assume that the interarrival time of successive items form a renewal process which can be characterized by the ditrsibution function of time between two successive points, A(x) with the finite mean $1/\lambda$, Laplace-Stieltijes Transform $A^*(\theta)$ and the probability density function a(x). The service mechanism consists of s servers, each of whose service time is negative expenontially distributed with the finite means $1/\mu_i$ for i^{th} server for $i=1, 2, \ldots, s$.

1. Switching Rules:

The operation of the switching rules can depend on the characteristics of the item in question (for example, the impatient items) or the characteristic of the system (for example, the finite capacity of the waiting room). Generally, there are three times where a decision is necessary:

*The Korea Institute for Defense Analyses

(a) At the point of arrival of an item to the system

(b) During the time interval of waiting of an item in the system

(c) At the point of the item joining a server and beginning service

Example 1.1. Some impatient items will leave the system while in queue. If this leaving item is lost to the system, then this rule will be called *reneging.*

Example 1.2. Consider the situation when there is more than one waiting line. An item in one waiting line is allowed to move to another line. This will be called *jockeying.*

The above two examples are the switching rules applied to the time interval (b).

Example 1.3. As an example of (c) consider the situation when there are idle servers. An item ready to proceed into service must choose a server (for example, by a given probability distribution) from among the idle servers.

Different kinds of switching rules and different kinds of waiting lines will give different systems, which form the subject matter of this paper. Furthermore, it is assumed that the switching rules depend on the state of the system, N(t). The decision rule at the point (c) is assumed to follow the rule given in the Example 1.3.

1.1. Entry Rule.

The *entry rule* is how an arriving item joins the system at the point (a). Therefore it depends on $N_a(t)$, the state random vector at the point of an arrival. There are two decisions to make: (i) the decision whether an arrival joins or not which is called the *balking rule,* and (ii) the decision which server (among idle servers) to join.

We now define the following two probabilities when an arrival finds the system in state n:

Def. 1.1. Let b(n) be the probability that an arriving item joins the system.

Def. 1.2. Given an arriving item joins the system, let $\gamma_i(n)$ be the probability that an arriving item joins the $_i$th server for $i=1,\dots,s$.

1.2. Reneging Rule.

Since we assume that any item in service will not renege, this rule depends only through the number of items in queue.

Def. 1.3. Let the random variable of the time to next renege when there are n items in queue be negative exponentially distributed with the mean $1/\alpha(n)$ for $n=1,2,\dots.$

The above definition alone can give information on a reneging rule sufficient to characterize the structure of the states, but insufficient to deal with the waiting time. Thérefore we need a more specific description of the reneging rule.

Def. 1.4. The reneging rule for the single-line (there is only on common waiting live) system is defined as follow.

(i) The reneging rule only applies to the item in queue.

(ii) The time to the renege for the item in the $_i$th position in queue is negative exponentially distributed with the mean $1/\alpha_i$ for $i=1,2,\dots.$

(iii) Every item in the system acts independently.

(iv) We assume that $\alpha_1 \leq \alpha_2 \leq \dots$

Remark 1.1. The independence described in (iii) of the above definition gives $\alpha(n)=\sum_{k=1}^{n}\alpha_i$ for $n=1\ldots$. Therefore the definition 1.4 is the special case of the definition 1.3 However, a system with $\alpha_1=0$ and $\alpha_2=1$ and another system with $\alpha_1=\alpha_2=\frac{1}{2}$ give the same $\alpha(2)=2$. This implies the insufficient information on the structure of the system by the Definition 1.3.

2. The Description of Systems.

2.1. *The Single-Line System with the Switching Rule (S.S.S.R.)*

In order to simplity notation for N(t) we introduce the following for the single-line system.

Def. 2.1. Label the servers $1,\ldots,s$. Let the set $S=\{1,\ldots,s\}$. Define S_r be the set of all subsets of S containing r elcments for $r=1,\ldots,s$.

Then by the above definition N(t) take values in the set o, s, s+1,... and also any elcmenent(x) of S_r, $r=1,\ldots,s-1$.

Def. 2.2. Let the seguence $\{b_n\}$ be a non-increasing sequence of non-negative real numbers where $b_n=b(s+n)$ for $n=0,1,\ldots$. Furthermore, $b(n)=1$ for all $n\epsilon Sr$, $r=1,\ldots,\ 0-1$, $b(\text{o})=1$, and $b_{-1}=0$. Then the vector $b=(b_0, b_1,\ldots)$ characterizes the balking rule. We call b the *balking probability vector.*

Since there is only one waiting line we only we only need to define $\gamma_i(n)$ when an arrival finds some set of idle servers.

Def. 2.3. Let $\gamma_i(\text{o})=\gamma_i$ for all $i\epsilon S$ and given $(x)\epsilon S_r$ for $r=1,\ldots,\ s-1$, we define

$$\gamma_i(n)=\gamma_i/\sum_{j\epsilon(x)j}\gamma \qquad if\ i\epsilon(x)$$
$$0 \qquad if\ i\epsilon(x)$$

where n is the state where the set (x) of servers are occupied.

Therefore two vectors b and $\gamma=(\gamma_1,\ldots,\gamma_\rho)$ complete the structure of the entry rule. We now define the S.S.S.R.

Def. 2.4. The system with the following structure will be called the S.S.S.R. (s, b, γ, α)

(i) There are s heterogeneous servers.

(ii) The entry rule has the two probabiity vectors b and γ.

(iii) The reneging rule follows the rule defined in the Defnaton 1.4.

(iv) There is one common waiting line.

Remark 2.1. *If* $\gamma_i=\frac{1}{s}$ for all i, the description of sets is necessary for the heterogeneous case, since in the homogeneous case there is no distinction between a set with γ occupied servers and another state with a different γ occupied servers.

2.2. The Multi-Line System with the Switching Rule (M.S.S.R.)

Let each server i have his own waiting room with capacity M_i for $i=1,\ldots,s$. Denote $\sum_{i=0}^{\rho}M_i$ by M. We consider systems which have some specific $\gamma_i(n)$ and $b(n)$.

Def. 2.5. The multi-line system with some $\gamma_i(n)$, $b(n)$ and a jockeying rule is called the M.S.S.R.

Now we give some examples of the M.S.S.R.

Example 2.1. Overflow System. (Ghiritis)

The multi-line system with the following properties will be called the *overflow system.*

(i) $b(n)=0$ if $n=(M_1,\ldots,M_s)$

1 otherwise

where M_i are finite for all $i=1,\ldots,s$.

(ii) $\gamma_i(n)=1$ if $n\epsilon\{n_i<M_i$ & $n_j=M_j$ for $i\leq i-1\}$

0 otherwise

(iii) Therer is no reneging and jockeying.

Example 2.2. Krishnamorthi system.

Let $M_2=0$ and $\rho=2$ The multi-line system with the following properties will be called the **Krishnamoorthi System.**

(i) $b(n)$ is the same as given in the Example 2.1.

(ii) There is a positive integer m ouch that $m<M_1$,

$\gamma\,(n)=1$ if $n,<m$

0 if $n,\geq m$

$\gamma_i(n)=1$ if $n,\geq m$

0 if $n,<m$

(iii) The item waiting in the m^{th} position performs instantaneous jockeying whenever the second server becomes available.

Remark 2.2 The above two examples were well studied by Ghiritis and Krishnamoorthi However, these are specific examples of the M.S.S.R. and can be more generalized.

3. Π, P and q Relations.

There are three instances which are freguently of interest. Let $t_0,t_1,\ldots,$ and $\tau_1,\ldots,$ be the points of arrival and departure ($t_o=0$).

However, in the systems which will be discussed here, there is a distinction to be made between arrivals which join and those which balk; we call both of them arrivals. Similarly a departure may not refer specifically to a service termination, but also to departures arising from balking and reneging.

Def. 3.1. The state probabilities are defined at the above instances according to the following notation.

$\Pi(n,\ t)=P_r(N(t)=n\,|\,t=t_m-$ for some $m)$

$P(n,\ t)=P_r(N(t)=n$ at an arbitrary time $t)$

$q(n,\ t)=Pr(N(t)=n\,|\,t=\tau_m+$ for some $m)$

Let the vectors $\Pi(t)$, $P(t)$ and $q(t)$ denote the corresponding probability vectors. If we drop t, then the respective values denote the steady state probabilities.

Natvig showed these differences for the general birth-death process.

3.1. Π-P Relations

The P distributions and Π distributions are usually derived from the balance eguations and the imbedded Markov chain (Kendall). When the input process is a Poisson process Cooper has given intuitive argument showing $\Pi(t)=P(t)$. We give a more analytic proof.

Thm 3.1. If the input process is Poisson process, then $P(t)=\Pi(t)$ for all $t\geq 0$.

Poof. The state of any system at any time point is completely determined by the sequence of arrivals and the service times prior to this point. Therefore, we have for any n

$$P_r(N(t)=n\,|\,t_0,\ldots,t_m\leq t,\ t_{m+1}>t)=P_r(N|t)=n\,|\,t_0,\ldots,t_m,\ t_{m+1}=t-)\ \text{for}\ m=0,\ldots$$

$$\Pi(n,\ t)=\sum_{m=0}^{\infty}P_r(N(t)=n\,|\,t_0,\ \ldots t_{m+1}=t-)Pr(t_0<t_1<\ldots<t_m<t\,|\,t_{m+1}=t)$$

$$P(n,\ t)=\sum_{m=0}^{\infty}P_r(N(t)=n\,|\,t_0,\ \ldots t_m\leq t,\ t_{m+1}>t)P_r(t_0<t_1<\ldots<t_m<t\,|\,t_m\leq t<t_{m+1})$$

However, $P_r(t_0<t_1<\ <t_m<t/t_{m+1}=t-)$

$$=\int_{0=t_0<t_1<\ldots<t_m<t}\frac{\lambda^{m+1}e^{-\lambda t}}{\frac{\lambda^{m+1}t^m e^{-\lambda t}}{m!}}dt_1dt_2\ldots dt_m=\int_{0=t_0<\ldots<t_m<t}\frac{m!}{t^m}dt_1\ldots dt_m$$

The uniform distribution of the order statistics from the Poisson process (Barlow and Proschan) gives

$$Pr(t_0<t_1\ldots<t_m<t\,|\,t_{m+1}=t-)=Pr(t_0<t_1<\ldots<t_m\leq t\,|\,t_m\leq t<t_{m+1})$$

Since n is any arbitrary state, we have $\Pi(t)=P(t)$.

We now discuss similar results in the non-Poisson case. Suppose there was an arrival at the time pointo- and let $V_1(t)$ be the time between the last arrival and time t, i.e., the post lifetime. Then the d.f. of $V_1(t)$ is known as follows (Cohen):

$$Pr(V_1(t)<x)=\begin{cases}0 & x\leq o\\ \int_{t-x}^{x}(1-A(t-u))dM(u) & o<x\leq t\\ 1 & x\geq t\end{cases}$$

where $M(u)$ is the renewal function of the input process.

The unpublished manuscript by Haight has given the p.d.f. of $V_1(t)$.

Lemma 3.2. The p.d.f. of $V_1(t)$ is

$$(1-A(x))\ (\delta(t-x)+m(t-x))\ \text{for}\ o<x\leq t$$

where $\delta(x)$ is a Dirac delta function and

$$m(x)=\frac{dM(x)}{dx}$$

Proof. $P_r(V_1(t)=t)=P_r(\text{no arrival during time t})=1-A(t)$.

Consider the case, $o<x<t$, then d.f. is differentiable. So, using the differentiation theorem,

$$\frac{dPr(V_1(t)<x)}{dx}=(1-A(x))m(t-x)$$

It is now necessary to consider a less general system. Specifically, consider a sysem where we can construct the *transition matrix function* $Q(x)$, whose elements are

$q_{m,n}(x)=Pr$(the transition occurs in time interval x to transform the system from the state m at t_K- *to* state n before t_{K+1} for any k, $k=0,1,\ldots$)

Then it is possible to form the imbedded Markov chain at the point of an arrival with the *transition matrix* Q, where

$$Q=\int_0^{\infty}Q(x)dA(x)$$

Denote the Laplace Transform (L.T.) of the functions $P(t)$, $Q(t)$ and $\Pi(t)$ by asterisk and also defime

$$\Pi_m^*(\theta)=\int_0^\infty e^{-\theta t}\Pi(t)m(t)dt$$

Next we need the fllowing two lemmas:

Lemma 3.3. Abelian Therem for L.S.T.

If a matrix function $Q(t)$ is of bounded variation in every finite interval and L.S.T. of $Q(t)$ is convergent for every $\theta>0$ and $\lim_{t\to\infty} Q(t)$ exists, then

$$\lim_{t\to\infty} Q(t)=\lim_{\theta\to 0}\int_0^\infty e^{-\theta t}dQ(t)$$

Proof See Widder.

Lemma 3.4. *Abelian Theorem for L.T.*

If a matrix function $Q(t)$ has a limit as $t\to\infty$ and L.T. of $Q(t)$ is convergent for every $\theta>0$, then

$$\lim_{t\to\infty} Q(t)=\lim_{\theta\to 0}\theta\int_0^\infty e^{-\theta t}Q(t)dt$$

Prnnf See Widder.

We can now prove for systems with the transition matrix function the following:

Theoren 3.5. $P^*(\theta)=P(o)Q_A^*(\theta)+\Pi_m^*(\theta)Q_A^*(\theta)$

Where $Q_A^*(\theta)$is L.S.T. of $Q_A(x)$,

$Q_A(x)=\int_0^x Q(y)(1-A(y))dy$ and $P(o)$ is the initial probability vector. Furthermore, if one of P, Π exists so does the other and

$$P=\lambda\int_0^\infty \Pi Q(y)(1-A(y))dy$$

Proof. The inital assumption that there was an arrival gives that the two initial probabilities are the same, i.e., $\Pi(o)=P(o)$.

By conditioning on the time $t-x$ of the last arrival we have

$$P(t)=\int_0^t \Pi(t-x)Q(x)dP_r(V_1(t)\le x)=\int_0^t \Pi(t-x)Q(x)(\delta(t-x)+m(t-x))(1-A(x))dx$$

by Lemma 3.2. Then by taking the L.T. and use the convolution theorem on L.T., we have

$$P^*(\theta)=\Pi(o)Q_A^*(\theta)+\Pi_m^*(\theta)Q_A^*(\theta)$$

Now, every integrable function is of bounded variation (Royden) Furthermor, the other conditions on $Q_A^*(\theta)$ for Lemma 3.3 are satisfied. Therefore the first term of the above equation vanishes as $t\to\infty$ and

$$\lim_{\theta\to 0} Q_A^*(\theta)=\int_0^\infty Q(y)(1-A(y))dy$$

If Π exists, then $\Pi(t)m(t)\to\lambda\Pi$ as $t\to\infty$ and

$$\Pi_m^*(\theta)\le\int_0^\infty e^{-\theta t}dm(t)=\frac{A^*(\theta)}{1-A^*(\theta)}$$

Lemma 3.4 implies that

$$\lim_{\theta\to 0}\theta\Pi_m^*(\theta)=\lambda\Pi$$

Thus

$$\lim_{t\to\infty}\int_0^t \Pi(t-x)Q(x)m(t-x)(1-A(x))dx=\lambda\Pi\lim_{\theta\to 0}Q^*(\theta)$$

implies that

$$\lim_{t\to\infty}P(t)=\lim_{\theta\to 0}\theta P^*(\theta)=\lambda\Pi\lim_{\theta\to 0}Q_A^*(\theta)$$

Similarly the exisence of Π can be shown if P exists.

Remark 3.1. It is known that the limiting *p.d.f.* of $V_1(t)$ is $\lambda(1-A(x))$. Therefore, an intuitive argument can be used to obtain the latter part of the above theorem.

Corollary 3.6. When $A(t)=1-e^{-\lambda t}$, then we have

$$P(t)=\Pi(t)$$

Proof. This corollary is a special case of Theorem 3.1. By conditioning on the time of the last arrival we have

$$m(t)\Pi(t)=\Pi(o)Q(t)a(t)+\int_0^t m(t-x)\Pi(t-x)Q(x)dA(x)$$

Since $m(t)=\lambda$ for all t, the above eguation reduces to

$$\Pi(t)=\Pi(o)Q(t)e^{-\lambda t}+\int_0^t \Pi(t-x)Q(x)\lambda e^{-\lambda t}dx$$

If we take L.T., then

$$\Pi^*(\theta)=\Pi(o)Q^*(\theta+\lambda)+\lambda\Pi^*(\theta)Q^*(\theta+\lambda)$$

Since $Q_A^*(\theta)=Q^*(\theta+\lambda)$ and $\Pi_m^*(\theta)=\lambda\Pi^*(\theta)$. The above theorem gives $P^*(\theta)=\Pi^*(\theta)$.

3.2. Π-q Relations.

There are very powerful results concerning Π-q relation (Cooper). One such result is that if we consider the states to be the total number of items and if there is a step function of an arrival and a departure by one, then $\Pi=q$. However the most of the systems with the switching rules have the vector valued states. Therefore this result is not necessarily true.

4. Supplementary Variable Method.

In this section we consider the S.S.S.R. (s, b, γ, α) with $\gamma_i=1/s$ for all i. It is possible to construct the transition matrix function $Q(x)$, but we use instead the supplmentary variable method used by Hokstad.

We consider the excess lifetime $U_2(t)$, *i.e.*, the time between t and the next arrival, as the supplementary variable.

Let $P_{(x)}(u,t)du=Pr(N(t)=(x),\ u\le U_2(t)\le u+du)$ for $(x)\epsilon S_r$, $r=1,2,\ldots,s-1$

$P_n(u,t)du=Pr(N(t)=n,\ u\le U_2(t)\le u+du)$ for $n=0,\ s,\ s+1,\ldots$

and let

$$P_r(u,\ t)=\sum_{(x)\epsilon S_r}P_{(x)}(u,\ t)\ \text{for}\ r=1,\ 2,\ldots,\ s-1$$

Define the steady-state probabilities if they exist by

$$P_n(u)=\lim_{t\to\infty}P_n(u,\ t)$$

$$P_{(x)}(u)=\lim_{t\to\infty}P_{(x)}(u,\ t)$$

Take L.T. (and denote it by an asterisk).

If P exists, then we have

$P_n = \lim_{\theta \to 0} P_n^*(\theta)$ and $P_{(x)} = \lim_{\theta \to 0} P_{(x)}^*(\theta)$

Theorem 4.1. The S.S.S.R. $(s, b, \gamma, \alpha,)$ with $\gamma_i = 1/s$ for all i has the following equations if P exists.

$$b_n P_{s+n}(o) = \nu_{n+1} P_{s+n+1} \text{ for } n=0, 1, \ldots$$

and

$$P_r(o) = B_r(o) \text{ for } r=0, 1, \ldots, s-1,$$

where $\nu_n = \mu + \alpha(n)$, $\mu = \sum_{i=1}^{s} \mu_i$, $\mu_{(x)} = \sum_{(a)\epsilon(x)} \mu_{(a)}$ and $B_r(\theta) = \sum_{(x)\epsilon S_r} \mu_{(x)} P_{(x)}^*(\theta)$

Proof. The proof essentially follows Hokstad's discussion with the added modification of balking and reneging for the states with more than s items and the heterogeneity for the case with less than s items.

The set of partial differential difference equations become

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) P_o(u, t) = \sum_{(a)\in s} u_{(a)} P_{(a)}(u, t) \quad (4\text{—}1)$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) P_{(x)}(u, t) = -\mu_{(x)} P_{(x)}(u, t) + \sum_{(a)\epsilon(x)} u_{(a)} P_{(x\cup a)}(u, t) + \frac{1}{s-r+1} \sum_{(a)\epsilon(x)} a(u) P_{(x-a)}(o, t) \quad (4\text{—}2)$$

for $(x)\epsilon S_r$, $r=1, \ldots, s-1$ and $(a)\epsilon S$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial u}\right) P_{n+s}(u, t) = -\nu_n P_{s+n}(u, t) + \nu_{n+1} P_{s+n+1}(u, t) + a(u) P_{s+n+1}(o, t) b_{n-1} + a(u) P_{s+n}(o, t)(1-b_n) \quad (4\text{—}3)$$

for $n=0, 1, \ldots$

The last two terms in equations (4—3) are from the fact that we consider seperately whether there is an item balking or not. Take L.T. after passing to the limit $t \to \infty$, then

$$\theta P_o^*(\theta) = P_o(o) - \sum_{(a)\epsilon s} u_{(a)} P_{(a)}^*(\theta) \quad (4\text{—}4)$$

$$(\theta - \mu_{(x)} P^*(\theta) = P_{(x)}(o) - \sum_{(a)\epsilon(x)} u_{(a)} P_{(x\cup a)}^*(\theta) - \frac{1}{s-r+1} \sum_{(a)\epsilon(x)} A^*(\theta) P_{(x-a)}(o) \quad (4\text{—}5)$$

for $(x)\epsilon S_r$, $r=1, \ldots, s-1$

$$(\theta - \nu_n) P_{n+s}^*(\theta) = P_{n+s}(o) - \nu_{n+1} P_{s+n+1}^*(\theta) - A^*(\theta)(b_{n-1} P_{s+n-1}(o) + (1-b_n) P_{n+s}(o)) \quad (4\text{—}6)$$

for $n=0, 1, \ldots$

Take the limit $\theta \to 0$ and then sum eguations (4—6) upto n, obtaining

$$\mu P_s = \nu_{n+1} P_{s+n+1} + P_{s-1}(o) - b_{n-1} P_{s+n}(o) \quad (4\text{—}7)$$

for $n=1, 2, \ldots$

Summing equations (4—5) for $(x)\epsilon S_r$, we have

$$\theta P_r^*(\theta) - B_r(\theta) = P_r(0) - A^*(\theta) P_{r-1}(0) - B_{r+1}(\theta) \quad (4\text{—}8)$$

for $r=1, \ldots, s-1$

Then equations (4—4) and (4—8) give

$$P_{s-1}(o) = B_s(o) = \mu P_s$$

Therefore from equation (4—7) we get the desired result

$$b_{n-1} P_{s+n}(o) = \nu_{n+1} P_{s+n+1} \text{ for } n=0, 1, \ldots$$

For the case when $N \leq s-1$, an inductive argument gives $P_n(o) = B_n(o)$.

Crollory 4. 2. Let Π_n and $\Pi_{(x)}$ be the corresponding values of P_n and $P_{(x)}$ respectively at the point of an arrival. Then

$$P_n(o)=\lambda\Pi_n \text{ for } n=0, 1,\dots$$

and

$$P_{(x)}(o)=\lambda\Pi_{(x)} \text{ for } (x)\epsilon S_r,\ r=1,\dots, s-1$$

Proof. Recall that the Π-distribution is a conditional distribution. Adding the equations (4—1) through (4—3), we have

$$\theta\sum_{i=0}^{\infty}P_i{}^*(\theta)=(1-A^*(\theta))\sum_{i=0}^{\infty}P_i(o)$$

and

$$\sum_{i=0}^{\infty}P_i{}^*(\theta)=1$$

give

$$\sum_{i=0}^{\infty}P_i(o)=\frac{1-A^*(\theta)}{\theta}=\lambda$$

Therefore the proof follows.

5. The Associated Systems.

Consider the single-line systems with different numbers of servers with the same b, γ, and α. Assume that all systms have the same total service rate μ. Then we call these the *associated systems.* Especially define the associated system with a single server as the *associated single-server system* (*A.S.S.S.*).

Denote the corresponding quantities for the number of servers by a number on the left side of the quantities, i.e.,

$${}^mP_{n+m}=P_r(N=n+m \text{ for the system with } m \text{ servers})$$

Then we have the following theorem.

Theorem 5. 1. Let

$${}^m\xi_n={}^m\Pi_{n+m}/\sum_{n=0}^{\infty}{}^m\Pi_{n+m} \text{ for finite } m\geq 1 \text{ and } n\geq 0$$

Then if ${}^m\xi_{n+m}$ exists for any m, it also does for every othery other m. Furthermore in this case

$${}^m\xi_n={}^1\xi_n \text{ for all finite } m \text{ and } n=0, 1,\dots$$

Proof. Choose an arbitrary m and make the partitions of ${}^m\Pi$ and ${}^1\Pi$ such that

$${}^m\Pi=({}^m\Pi_0,\ {}^m\Pi_1) \text{ and } {}^1\Pi=({}^1\Pi_0,\ {}^1\Pi_1)$$

where

$${}^m\Pi_1=({}^m\Pi_m,\ {}^m\Pi_{m+1},\dots)$$

and

$${}^1\Pi_1=({}^1\Pi_1,\ {}^1\Pi_2,\dots)$$

The transition matrix ${}^mQ$ has the submatrices ${}^mQ_i{}^1{}_s$ where

$${}^mQ=\begin{pmatrix}{}^mQ_1, & {}^mQ_2\\ {}^mQ_3, & {}^mQ_4\end{pmatrix}$$

Then ${}^mQ_4$ are the same for all m and

$${}^mQ_2=\begin{pmatrix} O & \\ A^*(\mu) & O\\ \vdots & \\ A^*(\mu) & \end{pmatrix}$$

where the number of $A^*(\mu)$'s is m.

Assume that ${}^1\Pi$ exist and decompose the eqnation ${}^m\Pi{}^mQ={}^m\Pi$ into two equations

$$ {}^m\Pi_0\,{}^mQ_1+{}^m\Pi_1\,{}^mQ_3={}^m\Pi_0 \qquad (5\text{—}1)$$

$$ {}^m\Pi_0\,{}^mQ_2+{}^m\Pi_1\,{}^mQ_4={}^m\Pi_1 \qquad (5\text{—}2)$$

Then ${}^m\Pi_1={}^md\,{}^1\Pi_1$ and ${}^m\Pi_{m-1}={}^md\,{}'\Pi_0$ satisfy the equation (5—1) for any finite positive ${}^md$. The non-singularity of the matrix $({}^mI-{}^mQ_1)$ gives the unique solution of ${}^m\Pi_0$ upto the constant multiple ${}^md$, and the normalizing condition gives the unique ${}^md$.

Assume ${}^m\Pi$ exist and multiply the equation (5—1) by ${}^m\epsilon=(1,\ldots,1)^T$ where T means the transpcse and the number $1's$ is 2^m-1. Then we have

$$ {}^m\Pi_0\,{}^mQ_1\,{}^m\epsilon=\sum_{i=0}^{m-1}\Pi_i-A^*(\mu)\,{}^m\Pi_m-1 $$

and

$$ {}^m\Pi_1\,{}^mQ_3\,{}^m\epsilon={}^m\Pi_1\,{}^1Q_3 $$

Let ${}^md'=\left(\sum_{i=0}^{\infty}{}^m\Pi_{m+i}+{}^m\Pi_{m-1}\right)^{-1}$, ${}^1\Pi_1={}^md'\,{}^m\Pi_1$ and ${}'\Pi_0={}^md'\,{}^m\Pi_{m-1}$. Then this ${}^1\Pi$ satisfies ${}^1\Pi{}^1Q={}^1\Pi$.

Since $\sum{}^1\Pi_i=1$, we have ${}^md=1/{}^md'$.

Therefore,

$$ {}^m\Pi_1=\frac{\sum_{i=0}^{\infty}{}^m\Pi_{ms}}{\sum_{s=0}^{\infty}{}^1\Pi_s}\,{}^1\Pi_1 \text{ and } {}^m\xi_n={}^1\xi_n. $$

Remark 5.1. The analysis of two classical systems $G|M|1$ and $G|M|s$ given in Karlin shows the same equilibrium conition $\lambda/\mu<1$ and the same geometric distribution as the conditional probabilities.

Remark 5.2. ${}^md$ is always less than 1 for any $m>1$. Therefore we have ${}^m\Pi_{n+m}<{}^1\Pi_{1+n}$ which implies the componentwise ineguality in the vector form ${}^m\Pi_1<{}^1\Pi_1$.

Let T be the limiting r.v. of the actual waiting time. Then

$$ Pr({}^mT\leq x\,|\,{}^mN_a=n+m)=Pr({}'T\leq x\,|\,{}'N_a=1+n) $$

where N_a represents the liniting r.v. of $N(t)$ given at the time of an arrival.

Corollary 5.2. $Pr({}^mT>x)=\left(\frac{{}^md}{{}^nd}\right)Pr({}^nT>x)$ for any $m,\ n>1$

Furthermore, $Pr({}^mT>x)>Pr({}^1T>x)$ for $m>1$.

Proof. The proof follows from the above remark.

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