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A Cash Management Model with Capital Gains Taxation-Two Assets Certainty Model

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Abstract

The cash management problem as a part of working capital management has been extensively studied. By and large the articles surveyed lacked consideration of long-term assets and the proper tax treatment of them. Recognizing that investment activities — long-term as well as short-term generate cash inflows, leads one to conclude that these investments should be included in the cash management problem. The liquidity of long term investments is an integral part of the cash management problem. This paper formulated a cash management models which incorporate the effect of long term investments and their liquidity on cash holdings.

Although all of the models formulated could be solved using mathematical programming techniques, the mere size of the problem in terms of the number of variables and constraints leads one to seek other methods. For this reason rules were developed using the Kuhn-Tucker conditions thereby substantially avoiding the programming calculations or at least easing them significantly.

I. Introduction

Every financial business has a cash management problem. Cash is normally held for both business transactions and for precautionary motives. On the other hand, the need for liquidity is balanced by the desire for profitability.

Models of the cash management problems typically studied (1, 8, 10, 12, 13, 15, 19) lead, according to Daellenbach (7), to little improvement in saving after they are applied. This may be due to neglecting the short-term implications of a long-term asset management problem. Most financial intermediaries are engaged in transactions with long-term marketable securities which are used to satisfy daily cash needs.

Furthermore even though some authors (2, 9, 14, 15) include a mix of long-term and short-term assets in their models they neglect the effects of the capital gains tax. A tax incentive to a long-term investmant may call for their inclusion of long-term assets into the cash managment model on the basis of providing for profitability as well as liquidity. Therefore cash in a view of a short-term liquidity should be extended to a view of a residual form of a portfolio management problem. In this respect, portfolio models with the consideration of liquidity needs and

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capital gains taxation will be studied in this paper.

Capital gains or loss frcm cpital assets are subject to a special treatment for tax-purposes. It is essential to include these special features in addition to ordinary income tax so that a more useful decision rule may be developed.

The tax rate on capital gains is determined not only by the length of the holding period but also by realized gains from capital assets. Allowable capital gains or loss resulting from its sale or exchange are restricted by tax regulations. The holding period of a capital asset determines the long-term or short-term character of gain or loss realized during the taxable year: an aasset is short-term if held for six months or less, and long-term if held for more than six months after purchase.

The models with capital gains taxation will include control variables of the form: s_{il} where s_{il} denotes the amount of an earning asset, in units, bought at time i and sold at time l. The control variables P_t and S_t, the amounts bought at time t and sold at time t, respectively, are $P_t=\sum_{l=t+1}^{T} s_{tl}$ and $S_t=\sum_{i=0}^{t-1} s_{it}$. All the control variables s_{il}'s are bounded by constants determined exogeneously by the investor. However, the investor may have a difficult situation where he can determine the upper bounds of P_t and S_t but he is unable to specify those of s_{il}'s. This situation will be solved by using a transportation algorithm to allocate the upper bounds of P_t and S_t to individual s_{tl} and s_{it}.

The following notation will be used in this paper:

G_{nl}, G_{ns}: not long-term and short-term capital gains, respectively

G_N: net capital gain $(G_{nl}+G_{ns}=G_N)$

TCG: total capital gains tax

d: normal income tax rate

d_1: long-term capital gains tax rate

Assume that $G_N>0$. Then

$$\text{TCG}=\begin{cases} d\cdot G_{ns}+d_1\cdot G_{nl} & \text{if } G_{ns}\geq 0 \text{ and } G_{nl}\geq 0 \\ d(G_{ns}+G_{nl}) & \text{if } G_{ns}\geq 0 \text{ and } G_{nl}\leq 0 \\ d_1(G_{ns}+G_{nl}) & \text{if } G_{ns}\leq 0 \text{ and } G_{nl}\geq 0. \end{cases}$$

II. Two Assets Certainty Model and Solution Approach

consider an investor with initial wealth $X_O+a_OZ_O$ (where X_O is cash and a_OZ_O is the value of Z_O shares of an earning asset) who wishes to maximize his total wealth, $X_T+a_TZ_T$, at time T. At any time t'ε\{t|t is an integer such that $0\leqq t\leqq T$\} the investor may buy and sell securities at known fixed prices a_t. Each transaction bears a cost proportional to the number of shares bought or sold. Transaction costs are paid immediately and serve to reduce cash. Proceeds from the sale of an earning asset will increase cash. The earning asset bears dividends.

The decision problem facing the investor is then

$$\max_{X_t,Z_t,P_t,S_t} \quad X_T+a_TZ_T-d\sum_{t=1}^{T}b_tZ_{t-1}-\text{TCG} \tag{2.1}$$

subject to

$$X_t=X_{t-1}-a_t\Big(\sum_{l=t+1}^{T}s_{tl}-\sum_{i=0}^{t-1}s_{it}\Big)-C_t\Big(\sum_{l=t+1}^{T}s_{tl}+\sum_{i=0}^{t-1}s_{it}\Big)+b_tZ_{t-1}-D_t,\ t=1,2,\ldots,T-1 \tag{2.2}$$

$$X_T = X_{T-1} + b_T Z_{T-1} - D_T \tag{2.3}$$

$$Z_t = Z_{t-1} + \sum_{l=t+1}^{T} s_{tl} - \sum_{i=0}^{t-1} s_{it}, \quad t=1, 2, \ldots, T-1 \tag{2.4}$$

$$Z_T = Z_{T-1} \tag{2.5}$$

$$\sum_{l=t+1}^{T} s_{tl}(=P_t) \leq M_{t\cdot} \text{ and } \sum_{i=0}^{t-1} s_{it}(=S_t) \leq M_{\cdot t}, \quad t=1, 2, \ldots, T-1 \tag{2.6}$$

$$s_{il} \geq 0, \quad i=0, 1, \ldots, l-1, \quad l=1, 2, \ldots, T \tag{2.7}$$

where $\mathrm{TCG} = d \cdot G_{ns} + d_1 \cdot G_{nl}$ for $G_{ns}, G_{nl} \geq 0$

$$= d \cdot \sum_{i=0}^{T-2} \sum_{l=i+1}^{i+5} (a_l - a_i)\, s_{il} + d_1 \sum_{i=0}^{T-7} \sum_{l=i+6}^{T-1} (a_l - a_i) s_{il} \text{ for } l \leq T-1 \tag{2.8}$$

Then we have the following Kuhn-Tucker conditions:

$$-d_1(a_K - a_t) + (a_K - C_K) + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) + u_{tk} - (v_{1t} + v_{2t}) = 0$$
$$\text{for } t+6 < K \leq T-1 \tag{2.9}$$

$$-d(a_K - a_t) + (a_K - C_K) + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) + u_{tK} - (v_{1t} + v_{2t}) = 0$$
$$\text{for } t < K \leq t+5 < T-1 \tag{2.10}$$

$$-(a_t + C_t) + a_T + (1-d) \sum_{i=t+1}^{K} b_i + u_{tT} - (v_{1T} + v_{2T}) = 0 \text{ for } K = T \tag{2.11}$$

$$u_{tK} s_{tK} = 0, \quad u_{tK} \geq 0, s_{tK} \geq 0 \tag{2.12}$$

$$v_{1t}(M_{t\cdot} - \sum_{l=t+1}^{T} s_{tl}) = 0, \quad v_{1t} \geq 0, \quad \sum_{l=t+1}^{T} s_{tl} \leq M_{t\cdot} \tag{2.13}$$

$$v_{2t}(M_{\cdot t} - \sum_{i=0}^{t-1} s_{it}) = 0, \quad v_{2t} \geq 0, \quad \sum_{i=0}^{t-1} s_{it} \leq M_{\cdot t} \tag{2.14}$$

Let (2.9), (2.10) and (2.11) be

$A_{tK} + u_{tK} - (v_{1t} + v_{2t}) = 0.$

Then the following decision rules are given:

If $A_{tK} < 0$ then $u_{tK} - (v_{1t} + v_{2t}) > 0$

and, therefore, 1) $u_{tK} > 0$ and $v_{1t} + v_{2t} > 0$,

2) $u_{tK} > 0$ and $v_{1t} + v_{2t} = 0$.

For all cases $s_{tK} = 0$.

If $A_{tK} = 0$ then $u_{tK} - (v_{1t} + v_{2t}) = 0$.

then 1) $u_{tK} = v_{1t} + v_{2t} = 0$

2) $u_{tK} = v_{1t} + v_{2t} > 0$.

For all cases s_{tK} is undetermined.

If $A_{tK} > 0$ then $u_{tK} - (v_{1t} + v_{2t}) < 0$.

then 1) $0 < u_{tK} < v_{1t} + v_{2t}$

2) $u_{tK} = 0 < v_{1t} + v_{2t}$

Then s_{tK} can be either positive or zero.

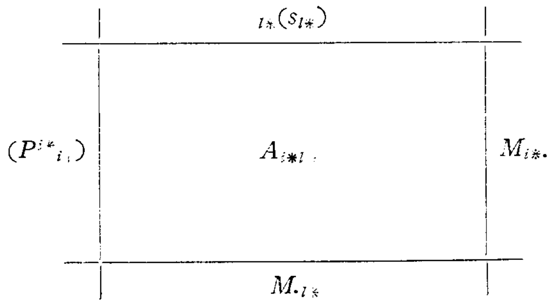
Therefore considering that the contribution of s_{tK} into the objective function is the corresponding marginal net profit A_{tK}, the following solution approach is suggested.

Step 1

For all $A_{tK} < 0$ set $s_{tK} = 0$ and eliminate the corresponding s_{tK} from a consideration.

Step 2

Form the following transportation problem for s_{i*l*} where the corresponding $A_{i*l*} \geq 0$:



Since $P_{i*}=\sum_{\text{all } l} s_{i*l}=M_{i*}.$ and $S_{l*}=\sum_{\text{all } i*} s_{i*l*}=M._{l*}.$

The solution of the transportation problem will be the optimal solution for the revised problem since it will satisfy the Kuhn-Tucker conditions (2.9)—(2.14).

The decision rules developed in Appendix are still useful for this problem when only the upper bounds of $\sum_{l=t+1}^{T} s_{tl}$ and $\sum_{i=0}^{t-1} s_{it}$ are available, since Step 1 for this problem is virtually the same as the decision rules in Appendix. The only difference is for $A_{tK}>0$ in the revised problem it is possible that $s_{tK}=0$ via Step 2 whereas this is not the case in Appendix. Therefore, first use the decision rules in Appendix and pick i^* and l^* where s_{i*l*} is either undtermined or positive. The corresponding s_{i*l*} will enter into the transportation problem in this problem.

III. Example and Conclusion

Consider an examlpe of the unexcluded dividend $b_t=.004$ dollars per share and the transaction cost $C_t=.01$ dollars per rhare for all t. Initial cash $X_0=500$ dollars and securities $Z_0=500$ shares. Upper bounds of P_t and S_t are $M_t.$ and $M._t$ shares, respectively, which will be defermined by the amounts of cash and securities available at time t, net returns at time t and other factors restricting the investors. Assume that $d=.5$ and $d_1=.25$.

$M_1.=160,\ M_2.=60,\ M_3.=100,\ M_4.=140,\ M.=80,\ M._6=120,\ M._7=80,\ M._8=160,$ $M._9=160,\ M._{10}=120$ and $M._{11}=80$.

Then the residual securities except $M_{11},\ _{12}$ are

$$Z_0+M_1.+M_2.+M_3.+M_4.-(M._5+M._6+M._7+M._8+M._9+M._{10}+M._{11})=160.$$

Market price in each period is qiven by the table 1.

〈Table 1〉 **Market Prices of an Asset**

time t	0	1	2	3	4	5	6	7	8	9	10	11	12
Market Price a_t	.9	.89	.91	.90	.88	.92	.93	.92	.94	.93	.91	.89	.90

Therefore the following transportation problem is formed:

	S_5	S_6	S_7	S_8	S_9	S_{10}	S_{11}	Residual at 12	$M_t.$
Z_0	0	.0145	.009	.026	.0205	.0075		.014	500
P_1	.003	.01	.0145	.315	.026	.013	0	.022	160
P_2				.145	.009			0	60
P_3		.001		.01	.0105	.0015		.008	100
P_4	.002	.009	.006	.018	.015	.0145	.0015	.026	140
$M._t$	80	120	80	160	160	120	80	160	

Entries of marginal profits are $A_{tK}'s$ which are from the solution process of the preblem in Appendix.

Optimal solution is

$s_{0,5}=80,\ s_{0,6}=120,\ s_{0,7}=80,\ s_{0,8}=100,\ s_{0,9}=120,$

$s_{1,9}=40,\ s_{1,10}=20,\ s_{1,11}=80,\ s_{1,12}=20,$

$s_{2,8}=60,$

$s_{3,10}=100$ and $s_{4,12}=140.$

which satisfies the assumption of net long-term and net short-term capital gains:

$$\sum_{i=0}^{K-7}\sum_{l=i+6}^{T-1}(a_l-a_i)s_{il}=19>0 \text{ and } \sum_{i=0}^{T-2}\sum_{l=i+1}^{i+t}(a_l-a_i)s_{il}=1.6>0.$$

Decision rules the model involve different capital gains tax rates which depend upon the number of periods the securities are held. The rules favor the long-term capital gains such that the marginal net profit is larger for long-term gains than one for the short-term gains by applying smaller tax rates if other things are equal.

For the simplest twe-asset deterministic model without taxes the result will shows simpler decision rules in contrast to those by Sethi and Thompson. The differences between the two models are:

(1) A dollar value of cash is re-evaluated period by period in their model while a dollar has a consistent value in this study.

(2) Security balance is epressed in terms of market value in their model while they are balanced in terms of the number of shares.

With the same objective function theirs depends upon the market prices up to the final period but in this study it was shown that only final market price is relevant in a decision.

Even for the model with capital qains taxes, this paper shows either simple decision rules or analytical way to solve the problem.

Appendix

Consider the same problem, (2.1)—(2.8) except that (2.6) is replaced by the following constraint:

$$s_{il}\leq M_{il},\ i=0,1,\dots,\ l-1,\ l=1,2,\dots,\ T. \qquad (A.\ 1)$$

Then the correponding Kuhn-Tucker conditions have v_{tK} instead of $v_{1t}+v_{2t}$, and instead of (2.13) and (2.14)

$$v_{tK}\ (M_{tK}-s_{tK})=0,\ v_{tK}\geq 0,\ s_{tK}\leq M_{tK} \qquad (A.\ 2)$$

The decision rules at time t, $t=1,2,\dots,\ T-1$ are easily established from (2.9)—(2.12) and (A. 2). For simplicity, let $t+6\leq K\leq T-1$ such that long-term capital gains are realized.

The decision rules are:

Case 1. *If* $-d_1(a_K-a_t)+(a_K-C_K)+(1-d)\sum_{i=t+1}^{K}b_i-(a_t+C_t)>0$, then $s_{tK}=M_{tK}$.

That this is true follows the fact that $u_{tK}-v_{tK}<0$ by $(2.9)\Rightarrow v_{tK}>0$ and $u_{tK}=0$ since u_{tK} and v_{tK} cannot both be positive. Then $s_{tK}=M_{tK}$ by (A. 2).

Case 2. *If* $-d_1\ (a_K-a_t)+(a_K-C_K)+(1-d)\sum_{i=t+1}^{K}b_i-(a_t+C_t)=0$, then $0\leq s_{tK}\leq M_{tK}$.

That this is true follows from the fact that $u_{tK}-v_{tK}=0$ by $(2.9)\Rightarrow v_{tK}=u_{tK}=0$ since u_{tK}

and v_{tK} cannot both be positive. Then $0 \leq s_{tK} \leq M_{tK}$ by (2.12) and (A. 2).

Case 3. $If - d_1\ (a_K - a_t) + (a_K - C_K) + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) < 0,$ then $s_{tK} = 0$.

That this is true follows from the fact that $u_{tK} - v_{tK} > 0$ by $(2.9) \exists v_{tK} = 0,\ u_{tK} > 0$ since $u_{tK}$and v_{tK} cannot both be positive. Then $s_{tK} = 0$ by (2.12).

Similarly, for $K = t+1, \ldots, t+5 \leq T-1$ at which time short-term gains are realized, the decision rules are as follows:

Case 1. $If - d(a_K - a_t) + (a_K - C_K) + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) > 0,$ then $s_{tK} = M_{tK}$.

Case 2. $If - d(a_K - C_t) + (a_K - C_K) + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) = 0,$ then $0 \leq s_{tK} \leq M_{tK}$.

Case 3. $If - d(a_K - C_t) + (a_K - C_K) + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t, < 0.$ then $s_{tK} = 0$.

Similarly, for $K = T$, the decision rules are as follows:

Case 1. If $a_T + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) > 0,$ then $s_{tT} = M_{tT}$.

Case 2. If $a_T + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) = 0,$ then $0 \leq s_{tT} \leq M_{tT}$.

Case 3. If $a_T + (1-d) \sum_{i=t+1}^{K} b_i - (a_t + C_t) 0,$ then $s_{tT} = 0$.

In words 1-3 can be stated as follows:

Case 1. If selling price plue dividends minus taxes and purchasing price, which is net marginal profit, is positive then buy and then sell the maximum allowed.

Case 2. If the net marginal profit is zero, then the amount of transaction is undetermined.

Case 3. If the net marginal loss is realized, then do nothing.

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