論 文
28—6—1

비연속 코스트를 갖는 최적 제어 문제의 필요충분조건

Necessary and Sufficient Conditions for an Optimal Control Problem Involving Discontinuous Cost Integrand

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Abstract

An optimal control problem in which the dynamics is nonlinear and the cost functional includes a discontinuous integrand is investigated. By using Neustadt's abstract maximum principle, a necessary condition in the form of Pontryagin's maximum principle is derived and it is further shown that this necessary condition is also a sufficient condition for normal problems with linear-in-the-state systems.

I. INTRODUCTION

One of the difficulties involved in applying the optimal control theory for practical problems is to define a suitable cost functional by which the performance of the controller is judged for optimality. In many cases, it is a difficult task to translate various requirements conceived by a designer into appropriate mathemetical expressions. On the other hand, if the cost functional is defined as desired, the resultant optimal problem may not be easily solved due to a nonstandard form of the cost functional. A class of nondifferentiable or vectorvalued cost functionals is a typical example.

In this paper, a norstandard cost functional of the form

$$J(\mathrm{u})=\int_{t_0}^{t_f} f^0(x(t),u(t),t)dt + \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} K_i(x(t),t)dt, \qquad (1.1)$$

where $t_0<t_1<t_2<\cdots<t_{N-1}<t_N=t_f$, is investigated. Note that the second term of $J(u)$ contains a discontinuous integrand. Cost functionals of this type seem appropriate for controlling a system in which a certain component of the state variable becomes of practical importance for minimization on a particular subinterval of the control duration. For example, in a regulator problem, an initial error, even if large, can be tolerated while errors after a certain period should be as small as possible. Also the cost functional such as above may be well-suited for long-term control problems involving biological systems, socioeconomic systems or political systems. For instance, an effective incumbent governor who wishes to get reelected seems to put particularly more efforts during the election year to improve the state of economy of the state.

In this note, a necessary condition for optimality is derived by using Neustadt's abstract maximum principle [1]. This necessary condition, which is valid for a general nonlinear system, is

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接受 日字：1979年 4月 11日

further shown to be a sufficient condition for optimality if the system under consideration is linear in the state and the cost integrands are convex in the state.

It. is remarked that in [2], Geering considered a cost functional including discrete state penalty terms of the form

$$J(u)=\sum_{i=1}^{N} H_i(x(t_i))+\int_{t_0}^{t_N} f^0(x(t),u(t),t)dt,$$

and presented a necessary condition. In this paper it will be shown that this result can be derived from ours as a special case.

In the sequel, the partial derivatives are sometimes denoted by subscripts

$$(e.g.\ f_x(x_0,u,t)=\frac{\partial}{\partial x}f(x,u,t)\Big|_{x=x_0}).$$

For a given matrx B, its transpose and the inverse are denoted B^T and B^{-1}, respectively. In general, column vectors are not distinguished. $z(\cdot)$, or simply z in some cases, denotes the function z(t) on an interval when it is regarded as an element of a function space.

II. PROBLEM FORMULATION

II-1. Problem Statement

Let $[t_0,t_N]$ be a given closed interval and $t_i\in R$ for, $i=1,2,\cdots$, N-1, be given such that $t_0<t_1<t_2<\cdots<t_{N-1}<t_N$. Consider the dynamic control system described by

$$\dot{x}(t)=f(x(t),u(t),t),\ t\in[t_0,t_N] \tag{2.1}$$

$$x(t_0)=x_0 \tag{2.2}$$

where x is an n-vector state variable and u is an r-vector control variable. f(x,u,t) and $\frac{\partial f}{\partial x}$ (x,u,t) are assumed to be continuous on $R^n x R^r x [t_0,t_N]$.

Let Ω be a given set in R^r. A measurable essentially bounded r- vector function $u(t)$ on $[t_0,t_N]$ is called **admissible** if $u(t)\in\Omega$ a.e. on $[t_0,t_N]$. U will denote the set of all admissible controls.

Let $\chi_i(x)$, $i=1,\ldots,m$, $f^0(x,u,t)$ and $K_j(x,t)$, $j=1,\ldots,N$ be given real-valued functions. It is assumed that $\chi_i(x)$ and $\frac{\partial \chi_i}{\partial x}(x)$ be continuous on R^n, $f^0(x,u,t)$ and $\frac{\partial f^0}{\partial x}(x,u,t)$ be continuous on $R^n\times R^r\times[t_0,t_N]$, and $K_j(x,t)$ and $\frac{\partial K_j}{\partial x}(x,t)$ be continuous on $R^n\times[t_{j-1},t_j]$.

The problem is to find an admissible control $u^*(t)$, $t\in[t_0,t_N]$ with response $x^*(\cdot)$ of Eqs. (2.1) such that the target condition

$$\chi_i(x^*(t_N))=0,\ i=1,2,\ldots,m \tag{2.3}$$

is satisfied and the cost functional

$$J(u)=\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} K_i(u(t),t)dt + \int_{t_0}^{t_N} f^0(x(t),u(t),t)dt \tag{2.4}$$

attains its minimum at $u(\cdot)=u^*(\cdot)$ (i.e., $J(u^*)\le J(u)$ for all $u\in U$ whose trajectory of (2.1) and (2.2) satisfies Eq. (2.3).)

II-2. Reformulation

Define a variable $x^0(t)$ by the relation

$$x^0(t)=\int_{t^0}^{t} f^0(x(s),u(s),s)ds.$$

Let $\hat{x}=(x^0,x)^T$

and $\hat{f}(\hat{x},u,t)=(f^0(x,u,t),f(x,u,t))^T$.

Let ε denote the set of all $(n+1)$-vector abseolutely continuous functions $\hat{x}(t)$ on $[t_0,t_N]$ such that

$$\dot{\hat{x}}(t)=\hat{f}(\hat{x}(t),u(t),t)\quad t\in[t^0,t_N]$$

for some $u(\cdot)\in U$. Define the functions $\phi^0:\varepsilon\to R^1$ and $\varphi^i\varepsilon\to R^1$, $i=1,2\ldots,m$ by

$$\phi^0(\hat{x}(\cdot))=x^0(t_N)+\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} K_i(x(t),t)dt$$

$$\varphi^i(\hat{x}(\cdot)),=\chi_i(x(t_N)),\ i=1,2,\ldots,m.$$

Then the problem stated in Sec. II-1 is equivalent to the following problem: Find an element $\hat{x}^*(\cdot)$ in ε such that

$$\varphi^i(\hat{x}^*(\cdot))=0,\ i=1,2,\ldots,m$$

and

$$\phi^0(\hat{x}^*(\cdot)\le\phi^0(\hat{x}(\cdot))\ \text{for all}\ \hat{x}(\cdot)\in\varepsilon.$$

Thus the original optimal control problem can be considered as a constrained minimization problem in a function space.

III. DERIVATION OF NECESSARY CONDITION

Let $u^*(t)$, $t\in[t^0,t_N]$, be an optimal control and $\hat{x}^*(t)$ on $[t^0,t_N)$ be the corresponding response. Then by letting $Z=R^1$, $Z=\{\gamma:\gamma<0\}$. $\varepsilon'=\varepsilon$, $\varphi=(\varphi^1,\varphi^2,\ldots,\varphi^m)$ and $\phi(\hat{x}(\cdot))=\phi^0(\hat{x}(\cdot))-\phi^0(\hat{x}^*(\cdot))$, one can easily see that $\hat{x}^*(\cdot)$ is a (φ,ϕ,Z)-extremal in the sense of Neustadt in [1]. Further, by choosing Y, M, h and $\hat{h}$ as indicated below, one can confirm the applicability of the abstract maximum principle in [1] for the problem under consideration. Specifically, let Y denote the space of continuous functions $\hat{x}:[t_0,t_N]\to R^{n+1}$ with the sup norm topology. Let $h=(h^1,\ldots,h^m)$: $Y\to R^m$ and $\hat{h}:Y\to Z$ be given by the formulas

$$h^i(\delta\hat{x}(\cdot))=\frac{\partial\chi_i(x^*(t_N))}{\partial x}\delta x(t^N),\ i=1,\ldots,m \quad (3.1)$$

$$h(\delta\hat{x}(\cdot))=D\phi^0(\hat{x}^*(\cdot):\delta\hat{x}(\cdot)) \quad (3.2)$$

for all $\delta\hat{x}=(\delta x^0,\delta x)\in Y$, where $D\phi^0(\hat{x}^*(\cdot):\delta\hat{x}(\cdot))$ denotes the Frechet differential of ϕ^0 at $\hat{x}^*(\cdot)$ with increment $\delta\hat{x}(\cdot)$ (see [3]). Finally let M be the convex hull of the $\{\delta\hat{x}_{\xi,u}(\cdot):\xi\in R^{n+1},\ u\in U\}$ where $\delta\hat{x}_{\xi,u}:[t_0,t_N]\to R^{n+1}$ is given by

$$\delta\hat{x}_{\xi,u}(t)=\hat{\Phi}(t)\xi+\hat{\Phi}(t)\int_{t_0}^{t}\hat{\Phi}^{-1}(s)[\hat{f}(\hat{x}^*(s),u(s),s)-\hat{f}(\hat{x}^*(s),u(s),s)]ds \text{ for } t\in[t_0,t_N]. \quad (3.3)$$

Here $\hat{\Phi}(t)$ denotes the fundamental matrix satisfying

$$\dot{\hat{\Phi}}(t)=\hat{f}_{\hat{x}}(\hat{x}^*(t),\ u^*(t),t)\hat{\Phi}(t) \quad (3.4)$$

$$\hat{\Phi}(t_0)=(n+1)\times(n\times1) \text{ identity matrix} \quad (3.5)$$

As in [1], one can show in a straightforward manner that $\hat{x}^*(\cdot)$ as an (φ,ϕ,Z) extremal satisfies the condition 3.1 in [1]. Therefore, from the abstract maximum principle of Neustadt (Theorem 3.1 in [1]), it follows that there exist numbers $\alpha^0,\alpha^1,\cdots,\alpha^m$ such that

$$\text{(i) } \alpha^0\le0,\ \sum_{i=0}^{m}|\alpha^i|>0 \quad (3.6)$$

$$\text{(ii) } \sum_{i=0}^{m}\alpha^i\frac{\partial\hat{\chi}_i}{\partial\hat{x}}(\hat{x}^*(t_N))\cdot\delta\hat{x}_{\xi,u}(t_N)+\alpha^0\sum_{i=0}^{N}\int_{t_{i-1}}^{t_i}\frac{\partial\hat{k}_i}{\partial\hat{x}}(\hat{x}^*(t),t)\ \delta\hat{x}_{\xi,u}(t)dt\le0$$

$$\text{for all } \xi\in R^{n+1},\ u\in U, \quad (3.7)$$

where $\hat{\chi}_i(\hat{x})=\chi_i(x)$ and $\hat{k}_i(\hat{x},t)=K_i(x,t)$. Set $\xi=0$ to obtain the relation

$$\int_{t_0}^{t_N}\sum_{i=0}^{m}\alpha^i\frac{\partial\hat{\chi}_i}{\partial\hat{x}}(\hat{x}^*(t_N))\hat{\Phi}(t^N)\hat{\Phi}^{-1}(s)[\hat{f}(\hat{x}^*(s),u(s)s)-\hat{f}(\hat{x}^*(s),s)]ds+\alpha^0\sum_{i=1}^{N}\int_{t_{j-1}}^{t_j}\frac{\partial\hat{k}_j}{\partial\hat{x}}(\hat{x}^*(t),t)\int_{t_0}^{t}\Phi(t)\Phi^{-1}(s)[\hat{f}(\hat{x}^*(s),u(s),s)-\hat{f}(\hat{x}^*(s),u^*(s),s)]ds\ dt\le0 \text{ for all } u(\cdot)\in U. \quad (3.8)$$

Define $\hat{\eta}_i(s):[t_0,t^N]\to R^{n+1}$, $i=1,2$, by

$$\hat{\eta}_1(s)=\left(\sum_{j=0}^{m}\alpha^j\frac{\partial\hat{\phi}^j}{\partial\hat{x}}(\hat{x}^*(t_N))\hat{\Phi}(t_N)-\alpha^0\int_{t_{i-1}}^{s}\frac{\partial\hat{k}_i}{\partial\hat{x}}(\hat{x}^*(t),t),\hat{\Phi}(t)dt\right)\hat{\Phi}^{-1}(s)$$

on $[t_{i-1},\ t_i]$, $i=1,\cdots,N$,

$$\hat{\eta}_1(t_N)=\hat{\eta}_1(t_{N-}),$$

$$\hat{\eta}_2(s)=\left(\alpha^0\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j}\frac{\partial\hat{k}_i}{\partial\hat{x}}(\hat{x}^*(t),t)\hat{\Phi}(t)dt\right)\hat{\Phi}^{-1}(s)$$

on $[t_{i-1},\ s_i]$, $i=1,\cdots,N$,

$$\hat{\eta}_2(t_N)=0.$$

Let $\hat{\eta}(s)=\hat{\eta}_1(s)+\hat{\eta}_2(s)$, $s\in[t_0,\ t_N]$.

Then $\hat{\eta}(t)$ is absolutely continuous on $[t_{i-1},t_i]$, $i=1,\ 2,\ N$, and

$$\dot{\hat{\eta}}(t)=-\hat{\eta}(t)\hat{f}_{\hat{x}}(\hat{x}^*(t),u^*(t),t)-\alpha^0\frac{\partial\hat{k}_i}{\partial\hat{x}}(\hat{x}^*(t),t) \quad \text{a.e. on } [t_{i-1},\ t_i],$$

$$\hat{\eta}(t_i^-)=\hat{\eta}(t_i^+)+\left(\alpha^0\int_{t_{i-1}}^{t_i}\frac{\partial\hat{k}_i}{\partial\hat{x}}(\hat{x}^*(t),\ t)\hat{\Phi}(t)ds\right)\Phi(t_i) \quad i=1,2\cdots,N.$$

$$\hat{\eta}(t_N)=\hat{\eta}(t_{N+})=\sum_{i=0}^{m}\alpha^i\frac{\partial\hat{\chi}_i}{\partial\hat{x}}(\hat{x}^*(t_N))-\left(\alpha^0\int_{t_{N-1}}^{t_N}\frac{\partial\hat{k}_N}{\partial\hat{x}}(\hat{x}^*(t),t)\hat{\Phi}(t)dt\right)\hat{\Phi}^{-1}(t_N).$$

Also using the general relation [4] of the form

$$\int_{t_{i-1}}^{t_i}\int_{t_0}^{t}g(t,s)ds\ dt=\int_{t_0}^{t_i}\int_{t_{i-1}}^{t_i}g(t,s)dtds-\int_{t_{i-1}}^{t_i}\int_{t_{i-1}}^{s}g(t,s)dt\ ds,$$

one can show after a short computation that the relation (3.7) implies the following integral form of maximum principle.

$$\int_{t_0}^{t_N}\hat{\eta}(t)[\hat{f}(\hat{x}^*(t),u(t),t)-\hat{f}(\hat{x}^*(t),u^*(t),t)dt\le0 \quad \text{for all } u(\cdot)\in U.$$

Since $\hat{f}$ and $\hat{k}_i$, $i=1,2,\cdots,N$ are independent of x^0, if $\hat{\eta}(s)\triangleq(\eta^0(s),\eta(s))$ with $\eta(s)\in R^n$, then

$$\dot{\eta}^0(t)=0$$

$$\eta^0(t_N)=\alpha^0.$$

Hence $\hat{\eta}(s)=(\alpha^0,\eta(s))$, and thus we obtain the following necessary condition.

Let $\Phi(t)$ be the $n\times n$ fundamental matrix such that

$$\dot{\Phi}(t)=f_x(x^*(t),u^*(t),t)\Phi(t)$$

$$\Phi(t_0)=(n\times n) \text{ identity matrix.}$$

THEOREM 1 (NECESSARY CONDITION):

Let $u^*(\cdot)$ with response $x^*(\cdot)$ be an optimal control. Then there exist numbers $\alpha_0, \alpha^1, \cdots, \alpha^m$ and piecewise absolutely continuous function $\eta(t)$ such that $\eta(t)$ is absolutely continuous on $[t_{i-1}, t_i]$, $i=1,2,\cdots,N$, and

(i) $\alpha^0 \leq 0, \ \sum_{i=0}^{m} |\alpha^i| > 0$

(ii) $$\dot{\eta}(t) = -\alpha^0 f_x{}^0(x^*(t), u(t), t) - \eta(t) f_x(x^*(t), u^*(t), t) - \alpha^0 \frac{\partial K_i}{\partial x}(x^*(t), t)$$ a.e. on $[t_{i-1}, t_i]$,

(iii) $$\eta(t_N) = \eta(t_N{}^+) = \sum_{i=1}^{m} \alpha^i \frac{\partial \chi_i}{\partial x}(x^*(t_N)) - \left(\alpha^0 \int_{t_{N-1}}^{t_N} \frac{\partial K_N}{\partial x}(x^*(t), t)\Phi(t)dt\right)\Phi^{-1}(t_N)$$

$$\eta(t_i{}^-) = \eta(t_i{}^+) + \left(\alpha^0 \int_{t_{i-1}}^{t_i} \frac{\partial K_i}{\partial x}(x^*(t), t), \Phi(t))dt\right)\Phi^{-1}(t_i)$$ $i=1,\cdots,N$,

(iv) $$\alpha^0 f^0(x^*(t), u^*(t), t) + \eta(t) f(x^*(t), u^*(t), t) = \max_{u \in \Omega} [\alpha^0 f^0(x^*(t), u, t) + \eta(t) f(x^*(t), u, t)],$$ a.e. on $[t_0, t_N]$.

Ⅳ. SUFFICIENCY RESULT

It can be shown that, under appropriate linearity and convexity asumptions indicated shortly, the necessary condition derived in Sec. Ⅲ is also sufficient for optimality for normal control problems. This sufficiency result, of course, shows the strength of the derived maximum principle in solving control problems.

The assumptions are as follows:

(A1) $f(x,u,t)$ is linear-in-the-state, i,e., $f(x,u,t) = A(t)x + g(u,t)$

(A2) $f^0(x,u,t)$ can be written as $f^0(x,u,t) = s^0(x,t) + c^0(u,t)$, and $s^0(x,t)$ is convex in x for each t.

(A3) $K_i(x,t)$ is convex in x for each t, $i=1,2,\cdots,N$.

(A4) $\chi_i(x)$ is affine in x, $i=1,2,\cdots,m$.

THEOREM 2 (SUFFICIENT CONDITION)

Let $u^*(\cdot)$ of Eqs. (2.1) and (2.2) which satisfies the target conditions Eq. (2.3). Suppose the problems data satisfies the assumptions (A1)~(A4). If there exist numbers $\alpha_0, \alpha^1, \cdots, \alpha^m$ with $\alpha^0 < 0$ and a function $\eta(t)$ on $[t^0, t_N]$ such that the conditions (ii)~(iv) of Theorem 1 are satisfied, then $u^*(\cdot)$ is an optimal control.

Proof:

Let $u(t)$ be an admissible control whose response $x(t)$ satisfies $\chi_i(x(t_N)) = 0$, $i=1,\cdots,m$.

Let

$$\Delta = \alpha^0 \int_{t_0}^{t_N} [f^0(x(t), u(t), t) - f^0(x^*(t), u^*(t), t)]dt + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} [K_i(x(t), t) - K_i(x^*(t)t)]dt.$$

It then suffices to show that $\Delta \leq 0$.

Recall that

$$x(t) - x^*(t) = \int_{t_0}^{t} [A(t)(x(s) - x^*(s)) + (F(u(s), s) - F(u^*(s), s))]ds$$

Since $\eta(t)$ is absolutely continuous on (t_{i-1}, t_i), $i=1,2,\cdots,N$, one may rewrite Δ as

$$\Delta = \alpha^0 \int_{t_0}^{t_N} [f^0(x(t), u(t), t) - f^0(x^*(t), u^*(t))]dt + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} [K_i(x(t), t) - K_i x^*(t), t)]dt + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i{}^-} d\eta(t) [x(t) - x^*(t)] - \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i{}^-} d\eta(t) \int_{t_0}^{t} [A(s)(x(s) - x^*(s)) + F(u(s), s) - F(u^*(s), s)]ds$$

It is a consequence of a result on Riemann-Stieltjes integral ([4. Theorem 5.5.7]) that

$$\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i{}^-} d\eta(t)[x(t) - x^*(t)] = \int_{t_0}^{t_N{}^-} \dot{\eta}(t)(x(t) - x^*(t))dt - \sum_{i=1}^{N-1} (\eta(t_i{}^+) - \eta(t_i{}^-))(x(t_i) - x^*(t_i))$$

Also, since χ_i is affine in x, one finds that

$$\eta(t_N{}^-)(x(t_N) - x^*(t_N)) = \sum_{i=1}^{m} \alpha^i \frac{\partial \chi_i}{\partial x}(x^*(t_N))(x(t_N) - x^*(t_N)) = \sum_{i=1}^{m} \alpha^i \left(\chi_i(x(t_N)) - \chi_i(x^*((t_N))\right) = 0,$$

$$\eta(t_0{}^+)(x(t_0) - x^*(t_0)) = 0.$$

Hence

$$\sum_{i=1}^{N}\eta(t)(x(t)-x^*((t)\Big|_{t_{i-1}^+}^{t_i^-}$$

$$=\sum_{i=1}^{N-1}(-1)\,(\eta(t_i^+)-\eta(t_i^-))(x(t_i)-x^*(t_i))$$

Therefore

$$\Delta=\alpha^0\left(\int_{t_0}^{t_N}[f^0(x,(t),u(t),t)-f^0(x^*(t),u^*(t),t)]dt\right.$$

$$\left.+\sum_{i=1}^{N}\int_{t_{i=1}}^{t_i}[K_i(x(t),t),t)-K_i(x^*(t),t)]dt\right)$$

$$+\int_{t_0}^{t_N^-}\eta(t)(x(t)-x^*(t))dt$$

$$-\sum_{i=1}^{N-1}[\eta(t_i^+)-\eta(t_i^-)](x(t_i)-x^*(t_i))$$

$$+\int_{t_0}^{t_N}\eta(t)\,[A(t)\,(x(t)-x^*(t))+(F(u(t),t)$$

$$-F(u^*(t),t))]dt$$

$$+(-1)\sum_{i=1}^{N}\eta(t)\,(x(t)-x^*(t))\Big|_{t_{i-1}^+}^{t_i^-}$$

Now make use of the conditions (ii) and (iii) to obtain

$$\Delta=\int_{t_0}^{t_N}\Big(\alpha^0c^0(u(t),t)+\eta(t)F(u(t),t)$$

$$-[\alpha^0c^0(u^*(t),t)+\eta(t)F(u^*(t),t)]\Big)dt$$

$$+\int_{t_0}^{t_N^-}\alpha_0\Big[s^0(x(t),t)-s^0(x^*(t),t)$$

$$-\frac{\partial s^0}{\partial x}(x^*(t),t)(x(t)-x^*(t))\Big]dt$$

$$+\alpha^0\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\Big[K_i(x(t),t)-K_i(x^*(t),t)$$

$$-\frac{\partial k_i}{\partial x}(x^*(t),t)(x(t)-x^*(t))\Big]dt\leq 0$$

The last inequality is an immediate consequence of the condition (iv) and the convexity assumptions on s^0 and K_i, $i=1,2,\cdots,N$.

Q.E.D.

V. CONCLUDING REMARK

The cost functional Eq. (1-2) considered by Geering in [2] may be rewritten as

$$\mathcal{J}(u)=\int_{t_0}^{t_N}f^0(x(t),u(t),t)dt$$

$$+\sum_{i=1}^{N}\lim_{h_i\to 0}\int_{t_i-h_i}^{t_i}\frac{H_i(x(t),t)}{h_i}dt$$

Since the cost functionals of the form

$$\mathcal{J}(u)=\int_{t_0}^{t_N}f^0(x(t),u(t),t)dt$$

$$+\sum_{i=1}^{N}\int_{t_i-h_i}^{t_i}\frac{H_i(x(t),t)}{h_i}dt$$

where $h_i<t_i-t_{i-1}$, $i=1,2,\cdots,N$, is a special case of (1.1), one may suspect that the result in [2] may be derived from Theorem 1 using the limiting process. In fact Theorem 1 of [2] can be derived from 1 of the present paper as a special case in a straightforward manner.

The theory presented in this note may be extended to the problem involving delay-differential systems, and in this case, the cost functional of the form Eq. (1.1) can used in incorporating a function target condition, say, $x(t)-\Psi(t)$ on $[t_f-h,t_f]$ as a part of the cost of the form $\int_{t_f-h}^{t_f}||x(t)-\Psi(t)||dt$.

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