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—Research Paper—

Convergence Properties of Iterative Methods for Linear Complementarity Problems

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Abstract

Convergence properties of a gereral iterative algorithm for linear complementarity problems (LCP's) are investigated. Iterative approaches to LCP's are mostly motivated by the large-scale and scarce problems. Convergence conditions are developed for general (the underlying matrix not necessarily symmetric) cases, and refined for several specific cases including the cases of Minkowski matrices and Quasi-dominant diagonal matrices.

1. INTRODUCTION

One area of the mathematical programming theory receiving increasing attention recently is the *complementarity programming theory*. Since the solution to a complementarity program is closely related to optimal solutions of many optimization problems, this complementarity programming theory constitutes a subbranch of the mathematical programming theory.

Even though the initial works in this area go back to the early 1960's, rigorous research e forts in theoretic and computational aspects of this area are relatively recent (see Cottle and Dantzig [5])

One of the most general formulations of the problem involves the following ingredients:

X: a real locally convex Hausdorff topological vector space,

Y: a real vector space,

F: a mapping from X to Y,

$\langle\cdot,\cdot\rangle$: a bilinear form on $X\times Y$,

K: a closed convex cone in X,

K^*: the polar of K in Y.

The *Complementarity Problem* (CP) is then to find a solution of the system

(1) $x\in K,\ F(x)\in K^*, \langle x, F(x)\rangle=0$

or, show that none exists.

While this statement of the problem is useful in calling attention to the search for function to the search for functionally related orthogonal vectors belonging to mutually polar cones, it is too general for numerical purposes. In nearly all cases, X and Y are the n-dimensional real space R^n, and $\langle\cdot,\cdot\rangle$ is the ordinary inner product there. Four major problem types can

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be identified. If the closed convex cone K is something more general than the nonnegative orthant, R^n_+, then (1) is referred to as a *generalized* complementarity problem (GCP). If the mapping F is an affine transformation, then (1) is known as a *linear* complementarity problem (LCP). Otherwise, it is a nonlinear complementarity problem (NLCP).

It can be shown without difficulty that linear programming, quadratic programming, and bimatrix (two-person, nonzero-sum) games lead to a LCP formulations. This implies that one solution algorithm for LCP can solve the above three distinct problems in a unified manner. In fact, one of the great accomplishments in the early history of LCP was the invention by Lemke and Howson [11] of a constructive algorithm for calculating a Nash equilibrium point for a bimatrix game. This work led to many significant including Lemke's method itself (called *almost-complementary pivoting* method). It is the foundation for algorithms that seek *economic equilibria.* See Scarf[16], Mathiesen [14], Dantzig, Eaves and Gale [8], and Ahn[1]. Many of economic equilibrium problems could be cast into NLCP rather than LCP.

Other applications of Complementarity programming include optimal stopping problems (Cottle[4]), engineering plasticity (Maier[12], Kaneko [10]), and some free-boundary problems (Cottle, Golub and Sacher[6]).

To date, most of the numerical work (development and analysis of algorithms) on complementarity problems has been concerned with the ordinary LCP, although some can be found on GNLP. The numerical methods for complementarity problems fall in two major categories: direct and iterative. *Direct Methods* are those based on the process of pivoting, that is, exchanging the roles of dependent and independent similar to basic and nonbasic variables (in LP) variables in a system of equations, while *iterative methods* are those which produce a (possibly infinite) sequence of iterates (trial solutions) which converge to a solution.

There are two main direct methods for solving the linear complementarity problem :

(2) $z \geq 0,\ Mz+q \geq 0, \langle z, Mz+q \rangle = 0$

where the cone $K=R^n_+$, and the mapping $F(z)=Mz+q$. One of these is the principle pivoting method (Cottle [3]), and the other is often called Lemke's almost-complementary pivoting method. Detailed review of these procedures is found in Cottle and Dantzig [5]. The conditions under which these numerical methods work much depend on the nature of the matrix M. Positive (Semi—) definiteness is a desirable matrix—theoretic property and one which is commonly found in applications, but not both of these methods are confined to this class of matrices. Other features are also of great importance. Some of these are the order of the matrix, its structure, its properties under pivotal transformation, and so forth. Even though the robustness of these direct methods, as far as the acceptable characteristics of the matrix M are concerned, is proven, their practicality is restricted due to the problem size limitation in computer implementations. Especially when it comes to the sparce but large matrix M, these direct methods are not well received since the pivoting steps involved in these algorithms destroy the sparcity of the matrix. On the other hand, the iterative methods typically modify the trial solutions of each iteration rather than the original problem data, leaving the sparcity of the matrix M all the way to the end of iterative procedure. This creates motivation for rigorous studies on iterative methods for LCP.

One of the oldest iterative methods related to LCP is due to Hildreth [9], who designed the procedure to solve a strictly convex quadratic program. Hildreth stated its Kuhn-Tucker conditions and used the nonsingularity of the Hessian matrix of the objective function to eliminate the primal variables. What remains after this operation is a LCP in which the variables are Lagrange multipliers and the matrix M is symmetric and positive semi-definite.

A more general iterative method attributed to Christopherson [2] has been analyzed and clarified by Cryer [7], and it is often cited as Cryer's method. It is a successive over-relaxation (SOR) method proposed for the solution of the free-boundaJy problem for journal bearings. It deals with the symmetric positive definite case.

Mangasarian [13] has introduced a robust iterative method for solviug LCP's. The algorithm has many options. Some of its realizations include a Jacobi-like algorithm, a generalization of the SOR method, and a generalization of the symmetric SOR (i. e. SSOR) method. The distinct assumption required for this method to be shown to converge is that the underlying matrix M should be symmetric.

This paper is similar to Mangasarian's work in its algorithmic aspects, but does concern with non-symmetric situation. This non-symmetric situations are not rare in economic eqilibrium problems, and are well exemplified in Ahn[1]. Main results of this work are the convergence properties of the non-symmetric problems when one of well-known iteration method is applied.

2. PRELIMINARIES.

We shall be concerned here with iterative methods for solving non-symmetric linear complementarity problem of finding z in R^n such that

(2) $z \geq 0, \ Mz+g \geq 0, \ z^T(Mz+g)=0,$

where M is a given $n \times n$ real non-symmetric matrix and q is a given $n \times 1$ vector. Non-symmetricity of M is not mandatory, since the algorithm and the theoritical results here could also be applied to a symmetric problem as a special case.

We briefly describe now the notation and some of the well-known results used in this paper. All matrices and vectors are real. A matrix A with m rows and n columns is said to be in $R^{m \times n}$. Row i of A is denoted by A_i, column j by $A \cdot$j, and the element in row i and column j by A_{ij}. If

$$I \subset \{1, \cdots, m\} \text{ and } J \supset \{1, \cdots, n\},$$

then A_{IJ} denotes the matrix extracted from A with elements A_{ij}, $i \in I$ and $j \in J$. The superscript T denotes the transpose. Superscripts, such as A^k, x^k, refer to specific matrices and vectors and usually denote iteration number. If $x \in R^n$, x_+ denotes the vector with elements

$$(x_+)_j = max\{0, x_j\}, \quad (j=1, \cdots, n)$$

If $x \in R^n$ and $A \in R^{n \times n}$, then

$$||x||^2_A = x^T A x,$$

$$||A||^2 = \text{spectral radius of } A^T A \equiv \rho(A^T A)$$

and

$$||x|| = (x^T x)^{\frac{1}{2}}$$

is the Euclidean norm. If $x \in R^n$, then the projection of x on R^n_+ is the unigue point in R^n_+ minimizing $||y-x||$ over all $y \in R^n_+$. It can be easily shown that the projection of x on R^n_+ is x_+. The M-matrices denote the class of square matrices with nonpositive off-diagonal elements and with a non-negative inverse, *i.e.*,

$$M_{ij} \leq 0, \text{ for all } i \neq j, \text{ and } M^{-1} \geq 0.$$

Lemma 1: Let $x, y \in R^n$ and A be a nonnegative $n \times n$ real matrix. Then,

(i) $x_+ - y_+ \leq (x-y)_+$,

(ii) $(x+y)_+ \leq x_+ + y_+$,

(iii) $x \leq y$ implies $x_+ \leq y_+$, and

(iv) $(Ax)_+ \leq Ax_+$

Proof: (i) There are four possibilities.

$x_i \geq 0,\ y_i \geq 0 \Rightarrow (x_+)_i - (y_+)_i = x_i - y_i \leq [(x-y)_+]_i$,

$x_i \geq 0,\ y_i \leq 0 \Rightarrow (x_+)_i - (y_+)_i = x_i \leq x_i - y_i \leq [(x-y)_+]_i$,

$x_i \leq 0,\ y_i \geq 0 \Rightarrow (x_+)_i - (y_+)_i = -y_i \leq 0 = [(x-y)_+]_i$,

$x_i \leq 0,\ y_i \leq 0 \Rightarrow (x_+)_i - (y_+)_i = 0 - 0 \leq [(x-y)_+]_i$.

(ii) and (iii) can be shown to be true similarly.

(iv) Since $x \leq x_+$ and $Ax \leq Ax_+$ (due to the nonnegativity of A), we get $(Ax)_+ \leq (Ax_+)_+ = Ax_+$. ///

These results are trivial but useful.

3. ITERATIVE ALGORITHM

We begin by stating and establishing a general fundamental iterative algorithm for solving the linear complementarity problem (2). As is well known that a fixed-point formulation readily lead to an iterative algorithm, the LCP (2) is transformed to the following fixed-point problem.

Lemma 2: Let $M \in R^{n \times n}$ and E be any positive diagonal martrix. Then,

$$\left.\begin{array}{l} Mz+g \geq 0 \\ z \geq 0 \\ z^T(Mz+g) = 0 \end{array}\right\} \Longleftrightarrow (z - \omega E(Mz+q))_+ - z = 0 \quad \text{for some or all } \omega > 0$$

The proof can be found in Mangasarian [13]. This is a neat result, and can be easily noticed that LCP (2) is transformed to a fixed-point problem of finding $z = f(z)$ where $f(z) = (z = \omega E(Mz+q))_+$.

This result result readily leads to the following general algorithm suggested by Mangasarian [13].

Algorithm 1: Let $z^o \geq 0$.

(3) $z^{k+1} = \lambda[z^k - \omega E^k(Mz^k + q + K^k(z^{k+1} - z^k)]_+ + (1-\lambda) z^k$, $k = 0, 1, 2 \cdots$, where $0 < \lambda \leq 1$, $\omega > 0$, and $\{E^k\}$ and $\{K^k\}$ are bounded seguences of matrices in $R^{n \times n}$, with each E^k being a positive diagonal matrix satisfying $E^k > \alpha I$ for some $\alpha > 0$ and the identity matrix I.

By setting $\lambda = 1, E^k = E$ and $K^k = K$ for each k, we obtain the following projected SOR algorithm.

Algorithm2: Let $z^0 \geq 0$.

(4) $z^{k+1} = [z^k - \omega E(Mz^k + q + K(z^{k+1} - z^k))]_+$, $k = 0, 1, \cdots$, where $w > 0$, E is a positive diagonal matrix, and K is a either upper or lower triangular matrix.

We first develop convergence criteria for this algorithm: and develop stronger results for some specific cases.

4. CONVERGENCE PROPERTIES

We state now the fundamental recursive inequality for the algorithm 2 which will serve as a basis for convergence and existence discussions later on. If $A \in R^{m \times n}$, then $|A|$ denotes the matrix obtained from A by replacing each element A_{ij} by its absolute value.

Lemma 3: Between the *k-th* and $k+1$-st solutions z^k and z^{k+1}, it follows that

(5) $|z^{k+1} - z^k| \leq (I - \omega E|K|)^{-1}(|I - \omega E(M-K)|)|z^k - z^{k-1}|$ for each k.

Proof: From (4),

$$z^{k+1} - z^k = [z^k - \omega E(Mz^k + q + K(z^k))]_+ - [z^{k-1} - \omega E(Mz^{k-1} + q + K(z^k - z^{k-1}))]_+$$
$$\leq [(z^k - z^{k-1}) - \omega EM(z^k - z^{k-1}) - \omega EK(z^{k+1} - z^k) + \omega EK(z^k - z^{k-1})]_+$$

Then, by Lemma 1. (iii) and (ii),

(6) $(z^{k+1} - z^k)_+ \leq [(I - \omega E(M-K))(z^k - z^{k-1})]_+ + [\omega E(-K)(z^{k+1} - z^k)]_+$.

Similarly, we can obtain the result for $z^k - z^{k+1}$, *i.e.*,

(7) $(z^k - z^{k+1})_+ \leq [(I - \omega E(M-K))(z^{k-1} - z^k)]_+ + [\omega E(-K)(z^k - z^{k+1})]_+$.

By noting that $|x| = x_+ + (-x)_+$ and by adding (6) and (7), we get

(8) $|z^{k+1} - z^k| \leq |I - \omega E(M-K))(z^k - z^{k-1})| + |\omega E(-K)(z^{k+1} - z^k)|$
$$\leq |I - \omega E(M-K)||z^k - z^{k-1}| + \omega E|K||z^{k+1} - z^k|.$$

Since K is a strictly upper or lower triangular matrix, the matrix $I - \omega E|K|$ is invertible, so that it follows that.

$$|z^{k+1} - z^k| \leq [I - wE|K|]^{-1}|I - wE(M-K)||z^k - z^{k-1}|. \quad ///$$

From this lemma, we can establish a condition for the sequence $\{z^k\}$ of the algorithm 2 to be bounded and has an accumulation point which solves the linear complementarity problem (2).

Theorem 1: Suppose that the given iteration parameters ω, E and K, and the underlying matrix M satisfy

(9) $\rho\{[I - \omega E|K]^{-1}|I - \omega E(M-K)|\} < 1$.

Then, the sequence $\{z^k\}$ of the algorithm 2 is bounded and has an accumulation point Z which solves the linear complementarity problem (2), and $\{z^k\}$ converges to Z. ×

Note that the condition (9) also provides an existence result for the LCP (2). Thus, when M satisfies (9) with properly chosen parameter values ω, E and K, the LCP(2) has a solution. By restricting parameter values to some fixed levels, we could idenitify the classes of matrices which satisfy the condition (9), and, accordingly, guarantee the existence of a solution to the LCP(2). Review of these special cases will be made in the next section.

Proof of Theorem 1: Let $G = [I - \omega E|K|]^{-1}|I - \omega E(M-K)|$.

Since $\rho(G)<1$ and $|z^{k+1}-z^k|\leq G|z^k-z^{k-1}|$ from lemma 3, we know that $\lim_{k\to\infty}|z^{k+1}-z^k|=0$.

Next we show the boundedness of the sequence.

Note that

$$
\begin{aligned}
|z^k-z^0| &\leq |z^k-z^{k-1}|+\cdots+|z^1-z^0| \\
&\leq (G^k+\cdots+I)\,|z^1-z^0| \\
&\leq (I-G)^{-1}|z^1-z^0| \quad (\text{from } \rho(G)<1) \\
&= \text{constant vector}
\end{aligned}
$$

for any k.

Then, by the Bolzano—Weierstrass Theorem, the sequence $\{z^k\}$ has a subsequence $\{z^{k_i}\}$ which converges to some limit (accumulation point), say z.

As a next step, we show that z solves the LCP (2). First, note that

$$\lim_{i\to\infty}|z^{k_i+1}-z|\leq|\lim_{i\to\infty}|z^{k_i+1}-z^{k_i}|+\lim_{i\to\infty}|z^{k_i}-Z|=0,$$

resulting in $\lim_{i\to\infty} z^{k_i+1}=z$.

Taking the limit operation on

$$z^{k_i+1}=[z^{k_i}-\omega E(Mz^{k_i}+q+K(z^{k+1}-z^{k_i}))]_+,$$

we get

$$z=[z-\omega E(Mz+q)]_+.$$

By lemma 2, then, z solves the LCP (2).

Finally, $\{z^k\}$ is shown to converge to z. Similar to the result of lemma 3, we get

$$|z^{k+1}-z|\leq G|z^k-z|,$$

where $G=[I-\omega E|K|]^{-1}[|I-\omega E(M-K)|]$. Since $\rho(G)<1$,

it follows from the numerical analysis theory that

$$\lim_{k\to\infty}|z^k-z|=0 \text{ or } \lim_{k\to\infty} z^k=z. \quad ///$$

Theorem 1 is the main result of this paper. Note that the matrix M is not assumed to be symmetric in the process of the above proof. To appreciate this result, we give in the following section some specific realizations of the fundamental algorithm 2, and identify the characteristics of M to satisfy the condition (9).

5. SPECIFIC ALGORITHMS AND CONVERGENCE RESULTS

Assume in this section that

$$M=L+D+U$$

where D is strictly lower triangular, and U is strictly upper triangular.

Theorem 2: Let the matrix M in the LCP (2) be a M-mtrix. Then, the seguence $\{z^k\}$ of the algorithm 2 with $K=L$ or U and with $\omega\leq 1/\max_j(D_{jj}E_{jj})$ converges to a solution of the LCP (2).

Proof: Since M is a M-matrix, EK and EM are also M-matrices. Notice also that

$$
\begin{aligned}
G&=[I-\omega E|K|]^{-1}[|I-\omega E(M-K)|] \\
&=[I+\omega EK]^{-1}[I-\omega E(M-K)]
\end{aligned}
$$

from $w \leq 1/\max_j (D_{jj}E_{jj})$. Then it follows that $\rho(G) < 1$ (see Ortega [15]), and Theorem 1 completes the proof. ///

The class of M-matrices is important in the linear complementarity programming theory, and has been studied by many (see Cottle [6], Cryer [7], for example). Cryer has worked on the convergence of the symmetric M-matrix case, and thus Theorem 2 is a more general result in that the matrix M is not restricted to be symmetric. It is to be noted that the algorithm of the above theorem is the *projected SOR* method, and that $E=D^{-1}$ and $\omega=1$ lead to the *projected Gauss-Seidel* method.

Theorem 3: Let M be a quasi-dominant diagonal matrix with positive diagonal elements, *i.e.*, for a positive diagonal matrix T,

$M_{ii}T_{ii} > \sum_{j \neq i} |M_{ij}T_{jj}|$ for each i.

Then the sequence $\{z^k\}$ generated by the algorithm 2 with $K=0$ and $w \leq 1/\max_j (M_{jj}E_{jj})$ converges to a solution of the LCP (2).

Proof: Here $G=|I-\omega EM|$. Consider the $l-\infty$ norm of

$|I-\omega T^{-1}EMT|$, *i.e.*, $\max_j [|1-\omega E_{ii}M_{ii}| + \sum_{j \neq i} \omega |E_{ii}M_{ij}T_{jj}/T_{ii}|]$.

Since $w \leq 1/\max_j (M_{jj}E_{jj})$ and $M_{ii}T_{ii} > \sum_{j \neq i} |M_{ii}T_{ii}|$, we have

$$|1-\omega E_{ii}M_{ii}| + \sum_{j \neq i} \omega |E_{ii}M_{ij}T_{jj}/T_{ii}|$$
$$< 1-\omega E_{ii}M_{ii} + \omega\ E_{ii}M_{ii}$$
$$= 1 \text{ for each } i.$$

Therefore, the $l-\infty$ norm $|||I-\omega T^{-1}EMT|||_\infty < 1$. Since

$$\rho(|I-\omega EM|) = \rho(|I-\omega T^{-1}EMT|) \quad \text{(similar matrices)}$$
$$\leq |||I-wT^{-1}EMT|||^\infty,$$

convergence is proved by Theorem 1. ///

Quasi-dominant diagonal matrices constitute an important class of matrices, too. The algorithm suggested for this class of matrices in Theorm 3 is the *projected Jacobi Overrelaxation*. Note further that $E=D^{-1}$, $\omega=1$ lead to the *projected* ordinary *Jacobi* iteration.

It might be possible to identify other classes of matrices which satisfy the condition (9) under appropriate specifications of iteration parameters ω, E and K. This remains as a future research topic.

References

[1] Ahn, Byong-hun, *Computation of Market Equilibria for Policy Analysis: the Project Independence Evaluation System (PIES) Approach*, Garland Publishing Inc., N.Y., New York, 1979.

[2] Christopherson, D.G., "A new mathematical method for the solution of film lubrication problems", *Inst. Mech. Engrs. J. Proc.* 146 (1941).

[3] Cottle, R.W., "The principle Pivoting Method of Quadratic Programming," in *Mathematics of the Decision Science*, Part I (G.P.Dantzig and A.F. Veinott, Jr., eds.), American Mathematical Socie-

ty, Providence, Rhode Island, 1968.

[4] ______________, "Numerical Methods for Complementarity Problems in Engineering and Applied Science," Technical Report no. SOL 77-24, Department of Operations Research, Stanford University, October. 1977.

[5] ______________, and G. P. Dantzig, "complementary Pivot Theory of Mathematical Programming", Linear Algebra and Appl. 1, 1968.

[6] ______________, G. H. Golub, and R. S. Sacher, "On the Sotulion of Large, Structured Linear Complementarity Problem," Technical Report no. 73-5, Department of Operations Research, Stanford University. 1973.

[7] Cryer, C. W., "The Solution of a Quadratic Programming Problem using Systematic Overrelaxation", *SIMA J. Control* 9(1971).

[8] Dantzig, G. B., B. C. Eaves, and D. Gale, "An Algorithm for a Piecewise Linear Model of Trade and Production with Negative Prices and Bankruptcy," Technical Report no. SOL 76-19, Department of Operations Research, Stanford University, 1976.

[9] Hildreth, "Point Estimates of Ordinates of Cancave Functions", *J. Amer. Statist. Assoc.* 49, 1954.

[10] Kaneko, I., "A Parametric Linear Complementarity problem involving derivative", Technical Report WP 77-23, Department of Industrial Engineering, University of Wisconsin-Madison, July 1977.

[11] Lemke, C. E., and J. T. Howson, Jr., Equilibrium Points of Bimatrix Games *J. Soc. Indust' Appl. Math.* 12, 1964.

[12] Maier G., "A Matrix Structural Theory of Piecewise Linear Elastoplasticity with Interacting Yield Planes", *Meccanica* 5, 1970.

[13] Mangasarian, O. L., "Solution of Symmetric Linear Complementarity Problems by Iterative Methods"., *J. Optim. Theory Appl.*, vol. 22, no. 4, August 1977.

[14] Mathiesen, L., "Efficiency Pricing in a Linear Programming Model: a case of constraints on the Dual Variables", Technical Report SOL 74-18, Department of Operations Research, Stanford University, 1974.

[15] Ortega, J. M., *Numerical Analysis, A Second Course*, Academic Press, N. Y., New York, 1972.

[16] Scarf, H., *The Computation of Economic Eqilibria*, Yale University Press, New Haven, Connecticut, 1973.