⟨Research paper⟩

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THE MINIMUM VARIANCE UNBIASED ESTIMATION OF SYSTEM RELIABILITY

C. J. Park*
Jae Joo Kim**

Abstract

We obtain the minimum variance unbiased estimate of system reliability when a system consists of n components whose life times are assumed to be independent and identically distributed either negative exponential or geometric random variables. For the case of a negative exponential life time, we obtain the minimum variance unbiased estimate of the probability density function of the i-th order statistic.

1. Introduction and Summary.

Suppose a system consists of n components and let X_i be the life time of the i-th component, $i=1, 2, \ldots, n$. The reliability of the system is defined by (see for example [1], page 20–49),

$$R(x, k, n) = P[\text{at least } (k+1) \text{ components are operative at time x}]$$
$$= P[\text{at least } (k+1)\ X_i\text{'s} > x],$$

where x is a fixed number and $k=0, 1, \ldots, n-1$. In this paper we obtain the minimum variance unbiased estimate of $R(x, k, n)$ when $X_1, X_2, \ldots X_n$ are independent and identically distributed geometric random variables with probability function $P[X_i=x]=(1-q)q^x, x=0, 1, \ldots, 0<q<1$, and when $X_1, X_2, \ldots X_n$ are independent and identically distributed negative exponential random variables with probability density function $f(x)=\lambda e^{-\lambda x}\ I_{(0,\infty)}(x)$. Notice that P[at least $(k+1)$ X_i's$>x$]$=1-$P[at least $(n-k)$ X_i's$\leq x$]. Thus in estimating the reliability $R(x,k,n)$, we consider the minimum variance unbiased estimate of the comulative probability function of $(n-k)$th order statistic $X_{(n-k)}$ induced by a random sample of size n. Let $T=\sum_{i=1}^{n} X_i$, then, in both cases, T is a sufficient statistic and the probability density function of T is complete since the probability function of T is an exponential family. Thus we use the Lehmann-Scheffé Theorem[3] and the Rao-Blackwell Theorem[4] to obtain the minimum variance unbiased estimate of $R(x, k, n)$. The reliability $R(x, n-1, n)=$P[at least n X_i's$>x$] can be viewed as the reliability of a series system and $R(x, o, n)=$P[at least one X_i's$>x$] may be viewed as the reliability of a parallel system. We also obtain the conditional probaility density function of the i-th order statistic $X_{(i)}$ given $T=t$ when X_i's are independent and identically distributed negative exponential random variables.

* Visiting Professor from the San Diego State University, San Diego, California, under SNU-AID Graduate Basic Science Program.
** Department of Computer Sciences and Statistics, Seoul National University.

2. Negative Exponential Life Time Distribution.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed with a probability density function given by

$$f(x)=\lambda e^{-\lambda x} I_{(0,\infty)}(x).$$

The reliability of the system can be written as

$$\begin{aligned} R(x,k,n) &= P[\text{at least } (k+1) X_i\text{'s}>x] \\ &= 1-P[\text{at least } (n-k)\ X_i\text{'s}\leq x] \\ &= 1-P[X_{(n-k)}\leq x], \end{aligned}$$

where $X_{(n-k)}$ is $(n-k)$th order statistic based on $X_1, X_2, \ldots, X_n$. Let $F(x)=P[X_i\leq x]=1-e^{-\lambda x}$, then the cumulative distribution function of $X_{(i)}$, the i-th order saatisics induced by $X_1, X_2, \ldots, X_n$, can be written

$$\begin{aligned} P[X_{(i)}\leq x] &= \sum_{j=i}^{n}\binom{n}{j}[F(x)]^j[1-F(x)]^{n-j} \\ &= \sum_{j=i}^{n}(-1)^j\binom{n}{j}\sum_{l=0}^{j}(-1)^l\binom{j}{l}[e^{-\lambda x}]^{n-l}. \end{aligned}$$

Now the statistic $T=\sum X_i$ is sufficient for λ and the family of the probability density function of T given by

$$f_T(t;\lambda)=\frac{\lambda^n t^{n-1}e^{-\lambda t}}{\Gamma(n)} I_{(0,\infty)}(t) \tag{1}$$

is a complete family. Thus from the Lehmann-Scheffé Theorem[3], there exists at most one estimate of $(e^{-\lambda x})^{n-l}$ as a function of T. If there is such an estimate then by the Rao-Blackwell Theorem[4], it will be the unique minimum variance unbiased estimate.

Let $g_n(x,t,l)$ be the minimum variance unbiased estimate of $(e^{-\lambda t})^{n-l}$ and set

$$(e^{-\lambda x})^{n-l}=\int_0^\infty g_n(x,t,l)\frac{\lambda^n t^{n-1}e^{-\lambda t}}{\Gamma(n)}dt. \tag{2}$$

But we can write

$$\begin{aligned} \Gamma(n)(e^{-\lambda x})^{n-l} &= \int_0^\infty \lambda^n t^{n-1}e^{-\lambda t}e^{-\lambda x(n-l)}dt \qquad (3) \\ &= \int_0^\infty \frac{t^{n-1}}{[t+x(n-l)]^{n-1}}\lambda^n[t+x(n-l)]^{n-1}e^{-\lambda(t+x(n-l))}dt. \end{aligned}$$

From (2) and (3)

$$\begin{aligned} &\int_0^\infty \frac{t^{n-1}}{[t+x(n-l)]^{n-1}}\lambda^n[t+x(n-l)]^{n-1}e^{-\lambda(t+x(n-l))}dt \\ &\quad =\int_0^\infty g_n(x,t,l)\ \lambda^n t^{n-1}e^{-\lambda t}dt. \end{aligned}$$

Consequently, we have

$$g_n(x,t,l)=\begin{cases}\left(\dfrac{t-x(n-l)}{t}\right)^{n-l} & \text{if } t>x(n-l) \\ 0 & \text{otherwise.}\end{cases}$$

Thus the minimum variased estimate of $P[X_{(i)}\leq x]$ is given by

$$\hat{P}[X_{(i)}\leq x]=\sum_{j=i}^{n}(-1)^j\binom{n}{j}\sum_{l=0}^{j}(-1)^l\binom{j}{l}\left[\frac{t-x(n-l)}{t}\right]_+^{n-1}.$$

where $[y]_+=y$ if $y\geq 0$ and 0 if $y<0$.

The minimum variance unbiased estimate of $R(x,k,n)$ is then

$$\hat{R}(x,k,n)=1-\hat{P}[X_{(n-k)}\leq x]$$
$$=1-\sum_{j=n-k}^{n}(-1)^{j}\binom{n}{j}\sum_{i=0}^{j}(-1)^{i}\binom{j}{i}\left[\frac{t-x(n-l)}{t}\right]_{+}^{n-1}. \quad (4)$$

It can be easily shown that

$$\hat{R}(x,o,n)=1-\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left[\frac{t-xk}{t}\right]_{+}^{n-1},$$

$$\sum_{j=0}^{n}(-1)^{j}\binom{n}{j}\sum_{i=0}^{j}(-1)^{i}\binom{j}{i}\left[\frac{t-x(n-l)}{t}\right]_{+}^{n-1}=1, \text{ and}$$

$$R(x,n-1,n)=\left[\frac{t-xn}{t}\right]_{+}^{n-1}$$

Note that $\left[\frac{t-xn}{t}\right]_{+}^{n-1}$ is the minimum variance unbiased estimate of $e^{-\lambda nx}$.

We can use the same technique to obtain the minimum variance unbiased estimate of the probability density function of $X_{(i)}$. The p.d.f. of $X_{(i)}$ is

$$h_n(y;\lambda,i)=\frac{n!}{(i-1)!(n-i)!}(1-e^{-\lambda y})^{i-1}(e^{-\lambda y})^{n-i}e^{-\lambda y},$$

and it can be shown that the minimum variance unbiased estimate of $h_n(y;\lambda,i)$ is given by

$$\hat{h}_n(y;\lambda,i,t)=\frac{n!}{(i-1)!(n-i)!}\frac{(n-1)}{t}\sum_{j=0}^{i-1}\binom{i-1}{j}\left[\frac{t-y(n+j+1-i)}{t}\right]_{+}^{n-2}.$$

Remark: (a) Notice that the joint probability density function of $Y_i=X_i/T, i=1,2\ldots, n$, is given by, (see[4]),

$$f(y_1,y_2,\ldots,y_n)=(n-1)!,\ \ y_i\geq 0 \text{ and } \sum_{j=1}^{n}y_j=1.$$

Let $Y_{(n)}$ by the $\max[Y_1,Y_2,\ldots,Y_n]$, then using (4) one can establish the following well known formula [2]

$$P[Y_{(n)}\leq y]=1-\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}(1-iy)_{+}^{n-1} \text{ or}$$

$$P[X_{(n)}>y]=\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}(1-iy)_{+}^{n-1}.$$

(b) Notice that $h_n(y;\lambda,i)$ is the conditional density function of the i-th order statistics $X_{(i)}$ given $T=t$. This follows from the fact that

$$h_n(y;\lambda,i)=\int_{-\infty}^{\infty}\hat{h}_n(y;\lambda,i,t)f_T(t;\lambda)dt,$$

where $f_T(t;\lambda)$ is given by (1).

3. Geometric Life Time Distribution

This is a discrete version of the results of section 2. Let $X_1,X_2,\ldots,X_n$ be independent and identically distributed with a probability function give by

$$p(x)=(1-q)q^x, x=0,1\ldots, 0<q<1.$$

Since $P[X_i>x]=q^{x+1}$, the reliability of the system can be written as

$$R(x,k,n)=\sum_{r=k+1}^{n}\binom{n}{r}(q^{x+1})^r(1-q^{x+1})^{n-r}$$

$$=\sum_{r=k+1}^{n}(-1)^r\binom{n}{r}\sum_{s=0}^{n-r}(-1)^{n-s}\binom{n-r}{s}(q^{x+1})^{n-s}.$$

The statistic $T=\sum_{i=1}^{n} X_i$ is sufficient for q, and the family of the probability function of T given by

$$P(t)=\binom{t-1}{n-1}(1-q)^n q^{t-n}$$

is a complete family. Now using the Rao-Blackwell Theoem it can be shown that the minimum variance unbiased estimate of $q^{(x+1)(n-s)}$ is

$$\hat{q}^{(x+1)(n-s)}=\begin{cases}\dfrac{\binom{t-(n-s)(x+1)-1}{n-1}}{\binom{t-1}{n-1}} & if\ t\leq n+(n-s)(x+1)\\ 0 & \text{otherwise.}\end{cases}$$

Thus the minimum variance unbiased estimate of $R(x, k, n)$ is given by

$$\hat{R}(x,k,n)=\sum_{r=k+1}^{n}(-1)^r\binom{n}{r}\sum_{s=0}^{n-r}(-1)^{n-s}\binom{n-r}{s}\hat{q}^{(x-1)(n-s)}$$

It can be easily shown that

$$\hat{R}(x,n-1,n)=\begin{cases}\dfrac{\binom{t-n(x+1)-1}{n-1}}{\binom{t-1}{n-1}} & if\ t\leq n+n(x+1)\\ 0 & \text{otherwise,}\end{cases}$$

$$\hat{R}(x,o,n)=1-\sum_{j=0}^{n}(-1)^j\binom{n}{j}\binom{t-j(x+1)-1}{n-1}\binom{t-1}{n-1},$$

where $\binom{t-j(x+1)-1}{n-1}=0\ if\ t-j(x+1)<n.$

Note that $\hat{R}(x,n-1,n)$ is the minimum variance unbiased estimate of $[q^{(x+1)}]^n$.

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