⟨Research Paper⟩

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Optimal Control of Dualistic Economic Growth

Sung Joo Park*

Abstracts

The paper illustrates a possible application of control theory to an economic growth system. Simultaneous nonlinear system of differential equations has been modeled which is different from the traditional formulation, based on the theory of economic growth for a two-sector (dual) economy. Necessary and sufficient conditions for the existence of the optimal control are derived directly from the Hamiltonian, and the optimal controls are also obtained by solving simultaneous equations. Obtaining the trajectories of the optimal control and state variables, however, should rely on the numerical procedures. Empirical applicaton has been conducted for the case of the Korean economy as an illustration.

I. Introduction

Recent trends in applying optimal control theory to the problem of socio-econcmic systems have been rooted to the question: "If optimal control theory can improve the guidance of airplanes and spacecraft, can it also be helpful in devising the policies for solving the problems of an economy or a society?" Even though the thoughts on this question are not new in the field of economic growth since the pioneering work of Ramsey [7], the theory thus developed over decades heavily relied on the balanced growth, golden-rule paths, or steady-state growth with little empirical applications to specific economies, thus creating a model of "mythical states" aloof from the reality [8].

The primary purpose of this paper is to demonstrate the practical usefulness of the theory of economic growth for the design of optimal economic policies with respect to the transitory behaviors by showing the optimal trajectories of an economy.

II. Dualistic Economic Growth Model

Central to developing a formal model of the dualistic model are the criteria employed in bisec-

* Korea Institute of Science and Technology.

ting the economy into analytically and empirically meaningful units. One possible framework for sectoral division was represented by the Meade and Uzawa's model [11], which is basically a straight extension of Solow's neoclassical one-sector model [10] by disaggregating an economy into an investment-goods sector and a consumption-goods sector. While this dichotomy may have some value in studying the equilibrium growth path of the industrialized economy, it is less useful for the less developed economy where the transitional phenomena between primitive and advanced (or agriculture and nonagriculture) dominate the long-run equilibrium paths.

If $K_i(t)$ and $L_i(t)$ denote the currently existing stocks of capital and labor for each sector, then the current rate of output of sector i, $Y_i(t)$, can be expressed by

$$Y_i(t)=A_i(t)F_i(K_i(t),\ L_i(t)) \tag{1}$$

where $A_i(t)$ is a measure of the current level of technical knowledge and $F_i(.)$ is the production function exhibiting certain characteristics. The total rate of output $Y(t)$ are disaggregated as

$$Y(t)=Y_1(t)+Y_2(t) \tag{2}$$

and if $C(t)$ and $S(t)$ denote the current rates of consumption and investment(assumed to be equal to saving), then from the national income identity

$$\begin{aligned} Y(t)&=C(t)+S(t) \qquad (3)\\ &=C_1(t)+C_2(t)+S_1(t)+S_2(t)\\ &=(1-s_1(t))Y_1(t)+(1-s_2(t))Y_2(t)+s_1(t)Y_1(t)+s_2(t)Y_2(t) \end{aligned}$$

where $s_i(t)$ is the fraction of current output of sector i that is being saved such that $0<s_i(t)<1$

The actual investment to each sector will be governed by an investment decision rule, $i(t)$, such that

$$I_1(t)=i(t)S(t) \tag{4}$$

$$I_2(t)=(1-i(t))S(t) \tag{5}$$

Where $I_i(t)$ are the gross investment for each sector and $i(t)$ is the investment-allocation parameter. If capitals of ecah sector are subject to evaporative decay at constant rates u_1 and u_2 respectively, the basic system state equations which govern the growth of an economy can be given as[1)]

$$\dot{K}_1(t)=I_1(t)-u_1K_1(t) \tag{6}$$

$$\dot{K}_2(t)=I_2(t)-u_2K_2(t) \tag{7}$$

where $K_i(t)$ represent the currently existing stocks of capitals for each sector. Using the equations (3) through (7), the system state equation can be written as

$$\dot{K}_1(t)=i(t)s_1(t)Y_1(t)+i(t)s_2(t)Y_2(t)-u_1K_1(t) \tag{8}$$

$$\dot{K}_2(t)=(1-i(t))s_1(t)Y_1(t)+(1-i(t))s_2(t)Y_2(t)-u_2K_2(t) \tag{9}$$

Retaining neoclassical assumptions on the production function, the system state equations can also be converted into per labor variables

$$\dot{k}_1(t)=-(u_1+g_1(t))k_1(t)+i(t)s_1(t)y_1(t)/l_1(t)+i(t)s_2(t)y_2(t)/l_1(t) \tag{10}$$

$$\dot{k}_2(t)=-(u_2+g_2(t))k_2(t)+(1-i(t))s_1(t)y_1(t)/l_2(t)+(1-i(t))s_2(t)y_2(t)/l_2(t) \tag{11}$$

In matrix form,

$$\begin{bmatrix}\dot{k}_1(t)\\ \dot{k}_2(t)\end{bmatrix}=\begin{bmatrix}-u_1-g_1(t) & 0\\ 0 & -u_2-g_2(t)\end{bmatrix}\begin{bmatrix}k_1(t)\\ k_2(t)\end{bmatrix}$$

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1) Dot denotes the time derivative of the variable.

$$+\begin{bmatrix} i(t)y_1(t)/l_1(t) & i(t)y_2(t)/l_1(t) \\ (1-i(t))y_1(t)/l_2(t) & (1-i(t))y_2(t)/l_2(t) \end{bmatrix}\begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} \tag{12}$$

or

$$\dot{\overline{k}}(t)=\underline{A}(t)\overline{k}(t)+\underline{B}(t)\overline{s}(t) \tag{13}$$

where

$y_i(t)=Y_i(t)/L(t)$

$k_i(t)=K_i(t)/L_i(t)$

$l_i(t)=L_i(t)/L(t)$

$g_i(t)=\dot{L}_i(t)/L_i(t)=(\dot{l}_i(t)+l_i(t)\dot{L}(t)/L(t))/l_i(t)$

The system described above is a time varying system. Moreover, since the output y_i is a function of the system state, thus making the coefficient matrix B a function of k, the system is nonlinear with coupling terms of state and input variables, which make the analytical solution impossible in general.[1]

Ⅲ. Optimal Economic Growth

The usulal economic growth problem of an economy can be posed as follows:

"What are the optimal program of capital accumulation which is consistent with the growth system equations and maximizes (or minimizes) a suitable criterion of a society while satisfying additional constraints?"

The social welfare or utility will be assumed to be a function of consumption and capital levels, and the objective functional will, then, be the integration of the social welfare over a time horizon with a discount rate such that[2]

$$\text{Max. } J(.)=\int_{t_0}^{t_f}U(\overline{c}(t),\ \overline{k}(t))e^{-rt}dt$$
$$=\int_{t_0}^{t_f}c_1(t)^{a_1}c_2(t)^{a_2}k_1(t)^{b_1}k_2(t)^{b_2}e^{-rt}dt \tag{14}$$

where $c_i(t)$ are the consumptions per labor of each sector, $[t_0, t_f]$ is the planning horizon, r is the rate of discount for the future utility, and U(.) denotes the social welfare or utility. Using the system state equations (12) or (13) and the objective functional (14), the Hamiltonian, $\hat{H}$, can be defined as

$$\hat{H}(.)=U(.)e^{-rt}+\hat{p}(t)^T(\underline{A}(t)\overline{k}(t)+\underline{B}(t)\overline{s}(t)) \tag{15}$$

where $\hat{p}(t)$ denotes the vector of costate variables and T stands for the transpose. For simplicity, it is convenient to define a modified Hamiltonian, H, as

$$H(.)=\hat{H}(.)e^{rt}=U(.)+\hat{p}(t)^Te^{rt}(\underline{A}(t)\overline{k}(t)+\underline{B}(t)\overline{s}(t))$$
$$=U(.)+\overline{p}(t)^T(\underline{A}(t)\overline{k}(t)+\underline{B}(t)\overline{s}(t)) \tag{15}$$

where $\overline{p}(t)$ denotes the vector of the modified costate variables.

1) Even for the simple case of linear production function, the resulting bilinear system doesn't have analytical solution in general.

2) The assumptions on the specific form of the social welfare function have been mute subjects. In fact, neither forms of the social welfare function can remove the basic conceptual difficulties, and the final choice is usually made on the basis of analytical convenience which satisfies a certain desirable conditions of utility.

The necessary conditions for the optimal control, then, can be derived as following:[1]

$$\dot{\bar{k}}=\frac{\partial H}{\partial \bar{p}}=\underline{A}\,\bar{k}+\underline{B}\,\bar{s}=\begin{bmatrix} v_{11}k_1+v_{12}y_1s_1+v_{12}y_2s_2 \\ v_{21}k_2+v_{22}y_1s_1+v_{22}y_2s_2 \end{bmatrix} \tag{17}$$

$$\dot{\bar{p}}=-\frac{\partial H}{\partial \bar{k}}+r\bar{p}$$

$$=-\begin{bmatrix} ((1-s_1)y_1)^{a_1}((1-s_2)y_2)^{a_2}k_1^{b_1}k_2^{b_2}(a_1y_1'/y_1+b_1/k_1) \\ +p_1(v_{11}+v_{12}y_1's_1)+p_2v_{22}y_1's_1 \\ ((1-s_1)y_1)^{a_1}((1-s_2)y_2)^{a_2}k_1^{b_1}k_2^{b_2}(a_2y_2'/y_2+b_2/k_2) \\ +p_1v_{12}y_2's_2+p_2(v_{21}+v_{22}y_2's_2) \end{bmatrix}+r\bar{p} \tag{18}$$

$$\bar{o}=\frac{\partial H}{\partial \bar{s}}=\begin{bmatrix} -a_1y_1((1-s_1)y_1)^{a_1-1}((1-s_2)y_2)^{a_2}k_1^{b_1}k_2^{b_2}+p_1v_{12}y_1+p_2v_{22}y_1 \\ -a_2y_2((1-s_1)y_1)^{a_1}((1-s_2)y_2)^{a_2-1}k_1^{b_1}k_2^{b_2}+p_1v_{12}y_2+p_2v_{22}y_2 \end{bmatrix} \tag{19}$$

where $v_{11}=-(u_1+g_1)$

$v_{12}=i/l_1$

$v_{21}=-(u_2+g_2)$

$v_{22}=(1-i)/l_2$

and $y_i'=dy_i/dt$.

The optimal control, $\bar{s}(t)$, can be obtained by solving the simultaneous equations (19),

$$s_1=1-\frac{1}{y_1}\left[\frac{k_1^{b_1}k_2^{b_2}}{\left(\frac{1}{a_1y_1}(p_1v_{12}y_1+p_2v_{22}y_1)\right)^{1-a_2}\left(\frac{1}{a_2y_2}(p_1v_{12}y_2+p_2v_{22}y_2)\right)^{a_2}}\right]^{\frac{1}{1-a_1-a_2}} \tag{20}$$

$$s_2=1-\frac{1}{y_2}\left[\frac{k_1^{b_1}k_2^{b_2}}{\left(\frac{1}{a_1y_1}(p_1v_{12}y_1+p_2v_{22}y_1)\right)^{a_1}\left(\frac{1}{a_2y_2}(p_1v_{12}y_2+p_2v_{22}y_2)\right)^{1-a_1}}\right]^{\frac{1}{1-a_1-a_2}} \tag{21}$$

where the saving rates are constrained between O and 1 under the normal economy.

The sufficient condition for the optimality can also be obtained from the Hessian matrix given by

$$\frac{\partial^2 H}{\partial \bar{s}^2}=a_1a_2y_1y_2((1-s_1)y_1)^{a_1-1}((1-s_2)y_2)^{a_2-1}k_1^{b_1}k_2^{b_2}$$

$$\begin{bmatrix} \frac{(a_1-1)y_1(1-s_2)y_2}{a_2y_2(1-s_1)y_1} & 1 \\ 1 & \frac{(a_2-1)y_2(1-s_1)y_1}{a_1y_1(1-s_2)y_2} \end{bmatrix} \tag{22}$$

The above Hessian matrix is negative definite if and only if

$$0<a_1,\ 0<a_2,\ \text{and } a_1+a_2<1 \tag{23}$$

The condition (23) guarantees the existence of global optimum at any instant of time, and hence the upper or lower limit of the constraint of control can be substuted to (20) and (21) whenever the unconstrained optimal saving rates violate the constraint. It can be observed that the sufficient condition for the optimality is only related to the exponents of consumption level but not those of capital.

1) For notational convenience, time t will be implicitly assumed for the remaining part of this section.

Ⅳ. Application to the Korean Economy: An Illustration

The Cobb-Douglas production functions are derived as following for the Korean economy:[1]

$$lnY_1(t)=-0.989519+0.393019\ lnK_1(t)+0.606981\ lnL_1(t)$$
$$lnY_2(t)=-0.506822+0.536155\ lnK_2(t)+0.463845\ lnL_2(t) \tag{24}$$

The system state equations, then, will become

$$\begin{bmatrix}\dot{k}_1(t)\\ \dot{k}_2(t)\end{bmatrix}=\begin{bmatrix}-1/25.5-g_1(t) & 0\\ 0 & -1/13.9-g_2(t)\end{bmatrix}\begin{bmatrix}k_1(t)\\ k_2(t)\end{bmatrix}$$
$$+\begin{bmatrix}(0.1)(0.371755)k_1(t)^{0.393019} & (0.1)(0.602407)k_2(t)^{0.536155}l_2(t)/l_1(t)\\ (0.9)(0.371755)k_1(t)^{0.393019}l_1(t)/l_2(t) & (0.9)(0.602407)k_2(t)^{0.536155}\end{bmatrix}\begin{bmatrix}s_1(t)\\ s_2(t)\end{bmatrix} \tag{25}$$

where the additional estimated values needed are

$$\dot{L}(t)/L(t)=0.03819 \tag{26}$$
$$l_1(t)=0.58504-0.01131t \tag{27}$$

and

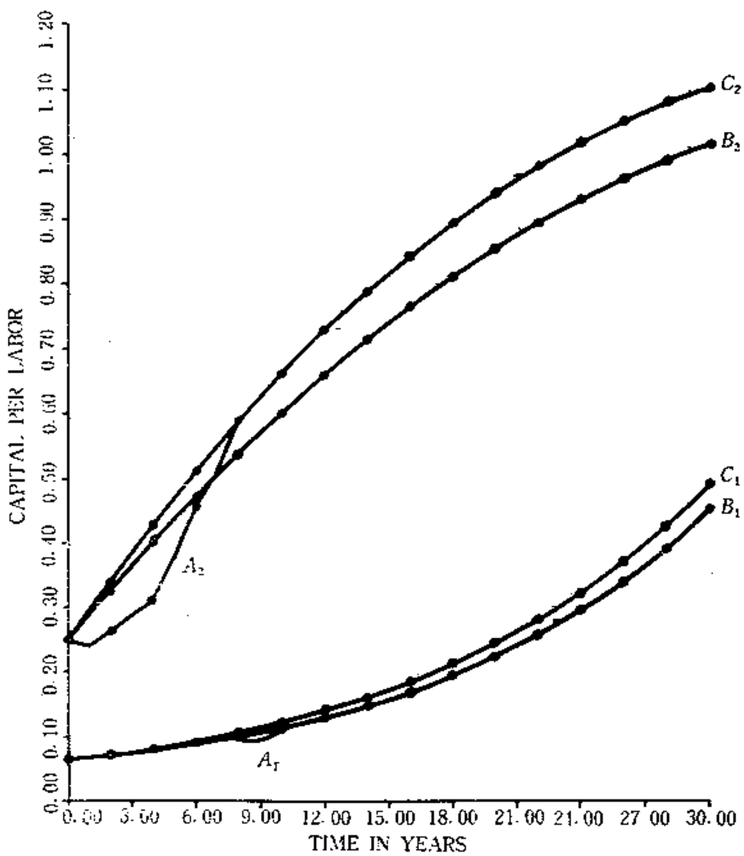

Figure 1 State Trajectories of Dual Economic Growth System

1) Detailed discussion on the data and the estimation results can be found in [5].

$$g_1(t)=(-0.01131+0.03819\ l_1(t))/l_1(t)$$
$$g_2(t)=(0.01131+0.03819\ l_2(t))/l_2(t) \tag{28}$$

To observe the responses of the system, step inputs, for an example, can be applied as following:

case B: $s_1(t)=0.05u(t)$

$s_2(t)=0.25u(t)$

case C: $s_1(t)=0.10u(t)$

$s_2(t)=0.25u(t)$

where $u(t)$ denotes the Heaviside step function. Case A is for the actual values. The trajectories

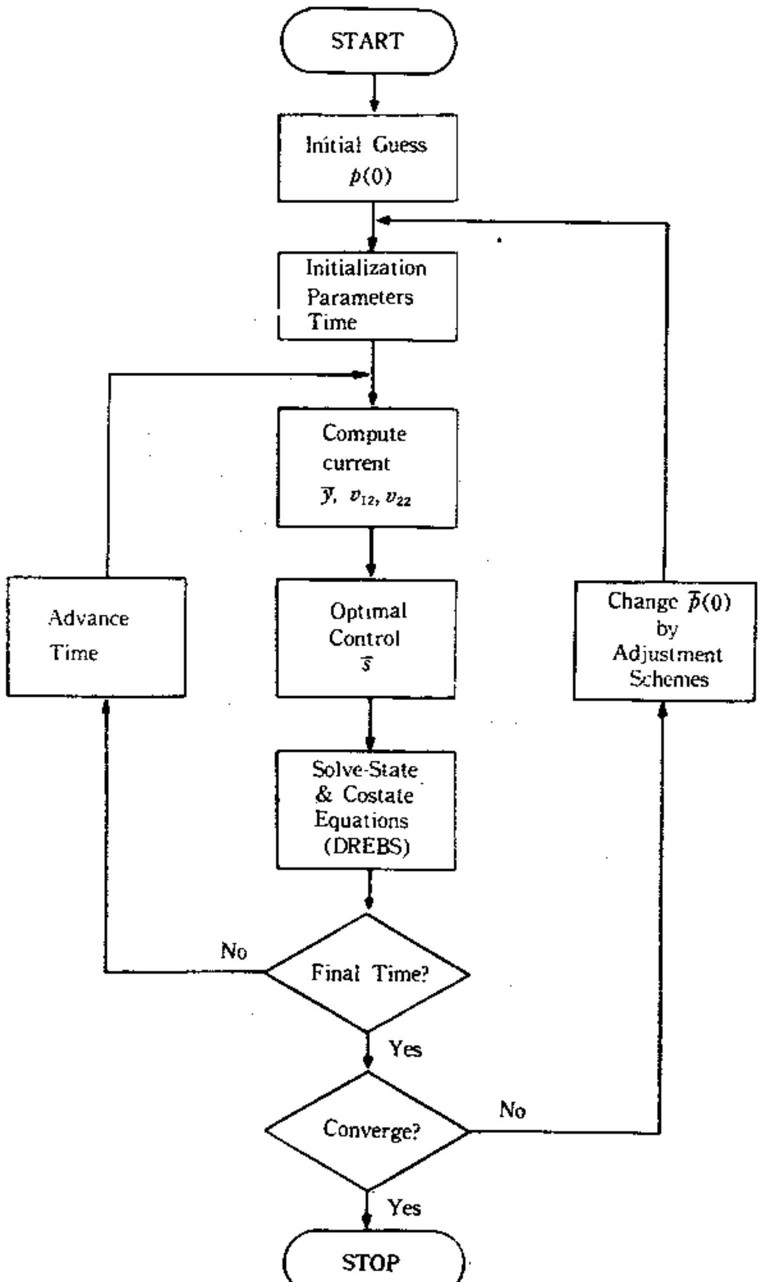

Figure 2 Numerical Procedure for the Optimal Trajectory of the Economic Growth System

of the three cases are plotted in Figure 1, where the subscripts denote the sectors 1 and 2.

In order to determine the optimal trajectories, the state and the costate equations should be solved simultaneously, which yields a nonlinear two-point boundary problem. Analytical solution of this kind is impossible in general, and thus necessarily relies on numerical procedures. A method based on variation of extremals shown in Figure 2. has been used in this paper, which can be justified by the linearity in the changes of the initial and the final costates. Adjustment scheme can be given by the costate influence function matrix such that

$$\overline{p}(t_0)_{new}=\overline{p}(t_0)_{old}-\underline{P}^{-1}\overline{p}(t_f) \qquad (29)$$

where the matrix $\underline{P}$ indicates the influence of changes in the initial costate on the costate trajectory at time t. For actual computations, the costate influence function matrix can be given as a diagonal matrix with constant elements between 0 and 1 for the convergence. Generally, the initial guess for $\overline{p}(t_0)$ is crucial to ensure the convergence, however, monotonically decreasing property of the costate function, which comes from the physical implication of the costates as the opportunity costs over the remaining time horizon, enables one to use the procedure free from the difficulty.

The results for the selected set of parameters—six cases denoted by arabic numbers—are given in Table 1, and the corresponding trajectories of the optimal control, the system state, costate, consumption, and total output are shown in Figures 3—7.

TABLE 1. RESULTS OF THE OPTIMAL CONTROL⁺

	1	2	3	4	5	6
$U(.)$‡	6.800	5.423	4.435	3.675	3.092	2.530
$y(10)$	0.5168	0.5874	0.6043	0.6116	0.6156	0.6182
$c(10)$	0.6513	0.7457	0.7685	0.7783	0.7838	0.7873
a_1, a_2	0.2	0.2	0.2	0.2	0.2	0.2
b_1, b_2	0.0	0.2	0.4	0.6	0.8	1.0
$k_1(10)$	0.3204	0.4014	0.4212	0.4297	0.4344	0.4374
$k_2(10)$	1.2040	1.5498	1.6395	1.6787	1.7008	1.7150
$p_1(0)$	1.1573	2.9047	3.7444	4.0659	4.0980	3.9785
$p_2(0)$	0.4127	0.8652	1.1186	1.2376	1.2782	1.2736
$p_1(10)$	-0.0047	0.0159	0.0146	0.0189	0.0134	0.0165
$p_2(10)$	0.0100	-0.0103	-0.0078	-0.0096	-0.0094	-0.0082
$\lvert\lvert p(10)\rvert\rvert^2$	0.0001	0.0004	0.0003	0.0005	0.0004	0.0003
ITER§	7	6	8	8	8	8

⁺Initial Conditions and Parameters:

$p_1(0)=2.5$, $p_2(0)=0.8$, ALPH=0.5, ERR=0.001

‡The value of U(.) is the value of social welfare function using the modified Hamiltonian, i.e., the value of non-discounted welfare.

§The number of iterations for the square Euclidean norm to be within the error limit (ERR).

As has been expected, the numerical algorithm with diagonal elements of the costate influence matrix equal to 1/2 shows fast convergence of the final costates to the origin. It can be observed that the change of saving rate from the upper limit to the lower limit occurs later as the relative weights on capitals are bigger, since the importance of the accumulation of capital has been increased. In other words, the inflections in the optimal trajectories occur when the society decides to save less and consume more, which can also be shown by the sudden rises of the consumption

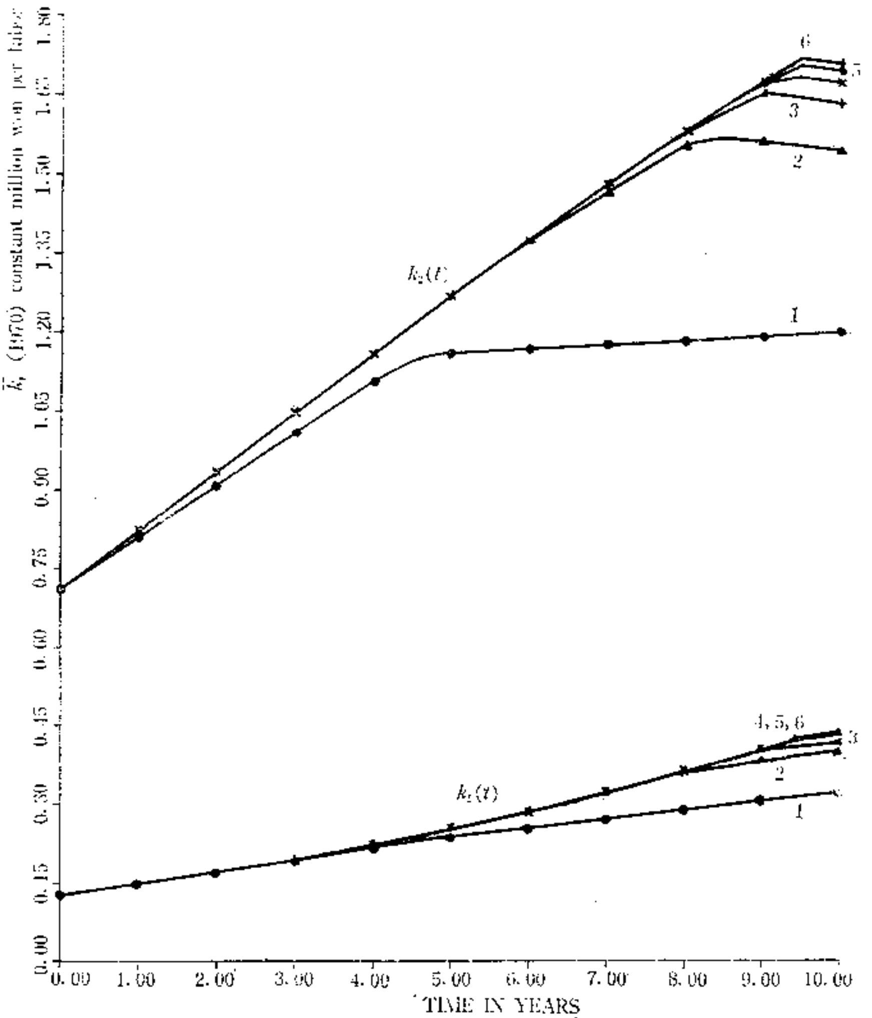

Figure 3 Optimal Trajectory. $\bar{k}(t)$

trajectories.

V. Concluding Remarks

The paper illustrated a possible application of optimal control theory to an economic growth system with specific assumptions on the objective functional and the structure of the dual economy. Illustration showed the existence of the optimal paths of an economy for the case of finite time horizon without constraints on the final states, which can easily be extended to the case with the final requirements.

The specific form of the social welfare has been used, in fact, to avoid the possible singularities in the case of the "bang-bang" control. Non-saturated controls, which may be preferrable to the saturated controls shown in Figure 5, may be obtained for the alternative combinations of

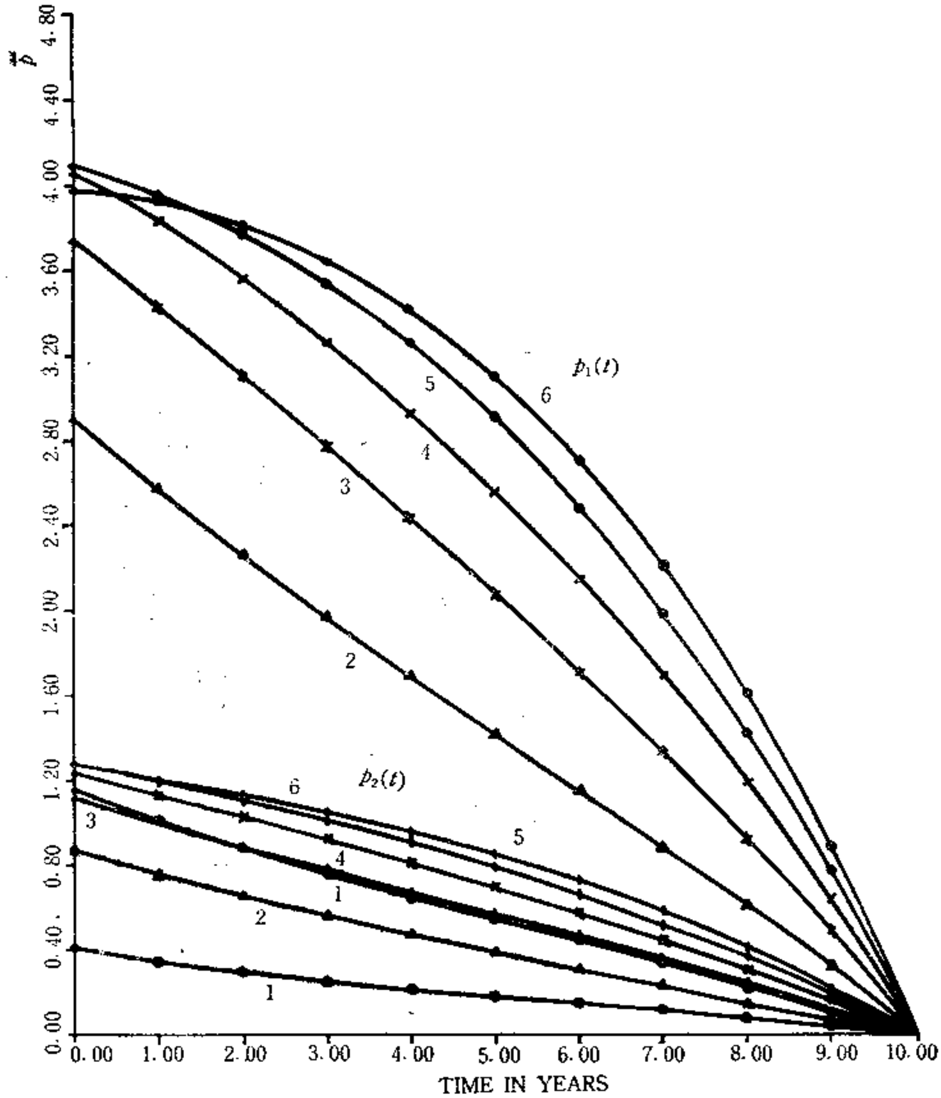

Figure 4. Costate Trajectory, $\bar{p}(t)$

relative weights of consumption and capital. The optimal saying schedule thus obtained may not be implemented directly, however, it can serve as a reference for the actual combined policies of monetary and fiscal measures, which can only be validated through the interactions with the decision makers. Further researches may include the forms of production function—including efficient labor, capital, and technical progress—, alternative saving functions, and investment strategies.

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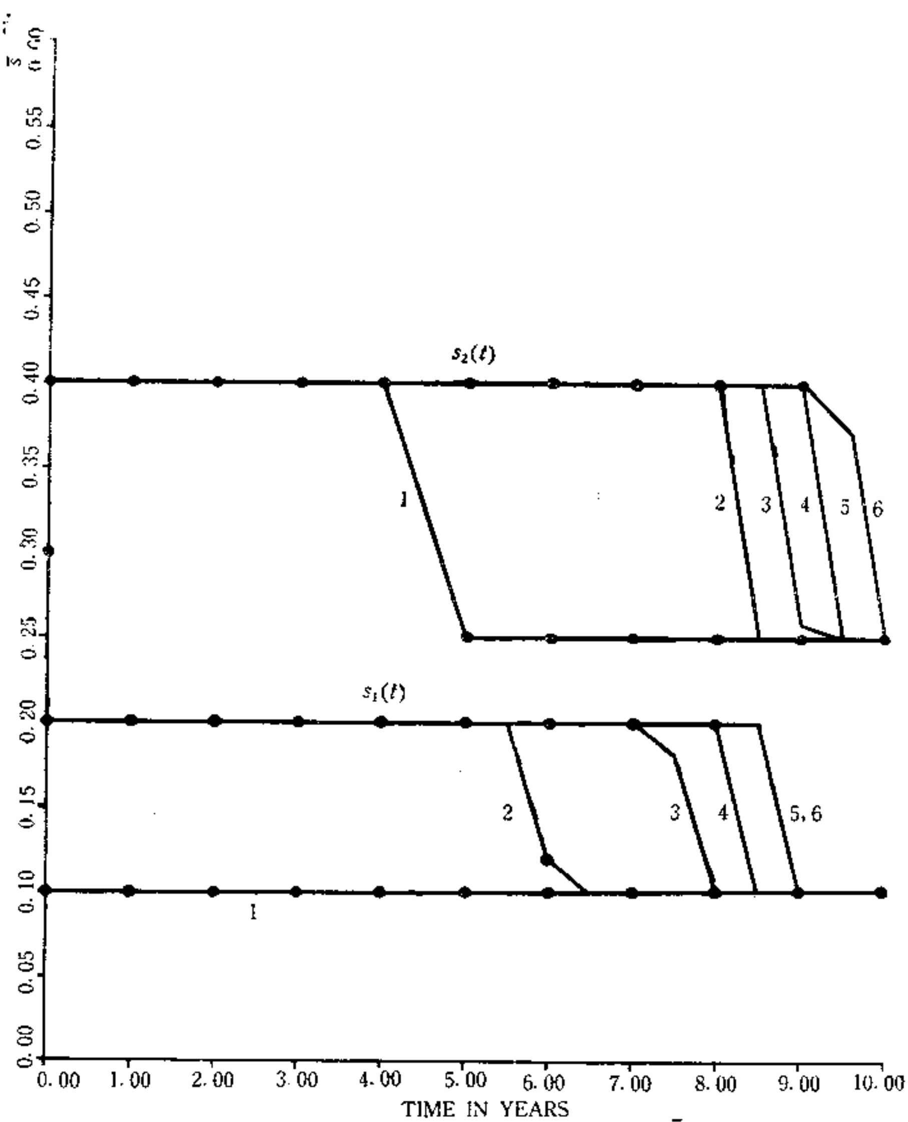

Figure 5. Optimal Control, $\bar{s}(t)$

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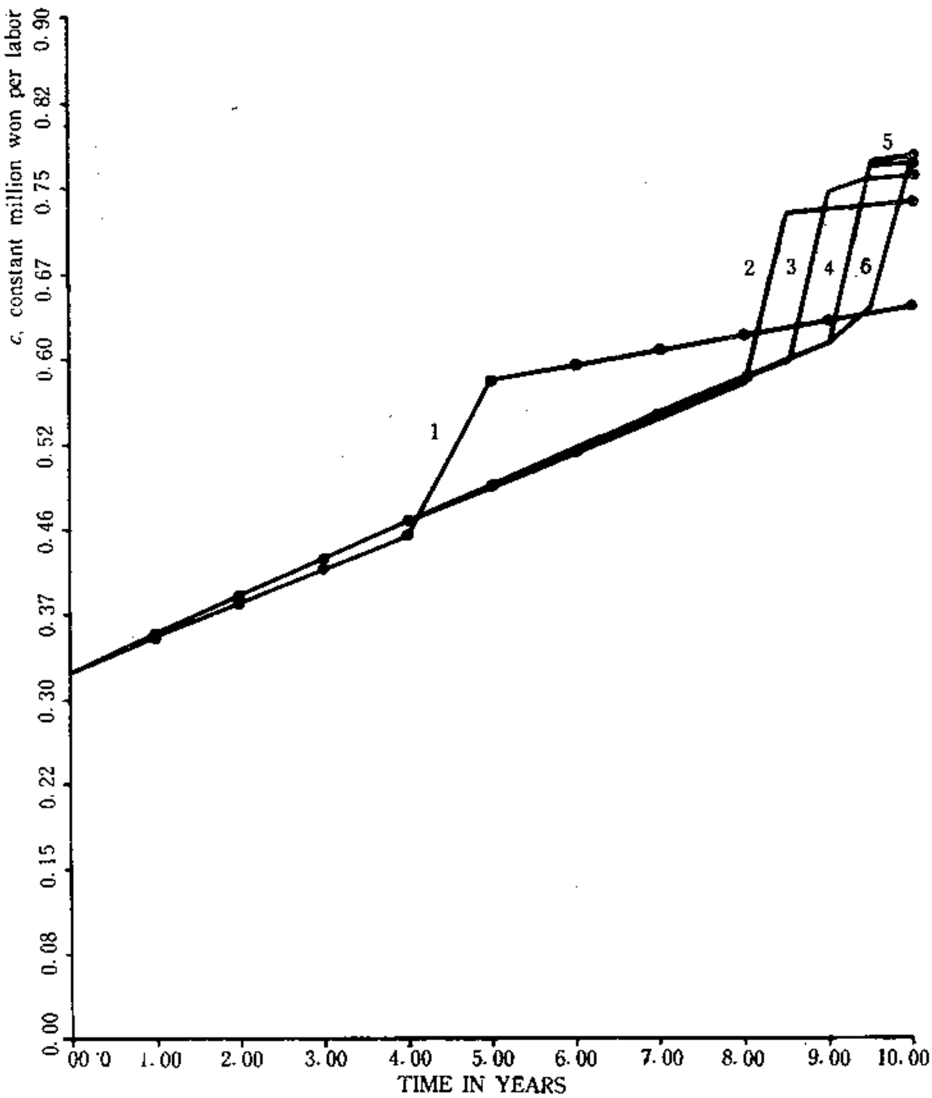

Figure 6. Trajectory of Consumption per Labor, $c(t)$

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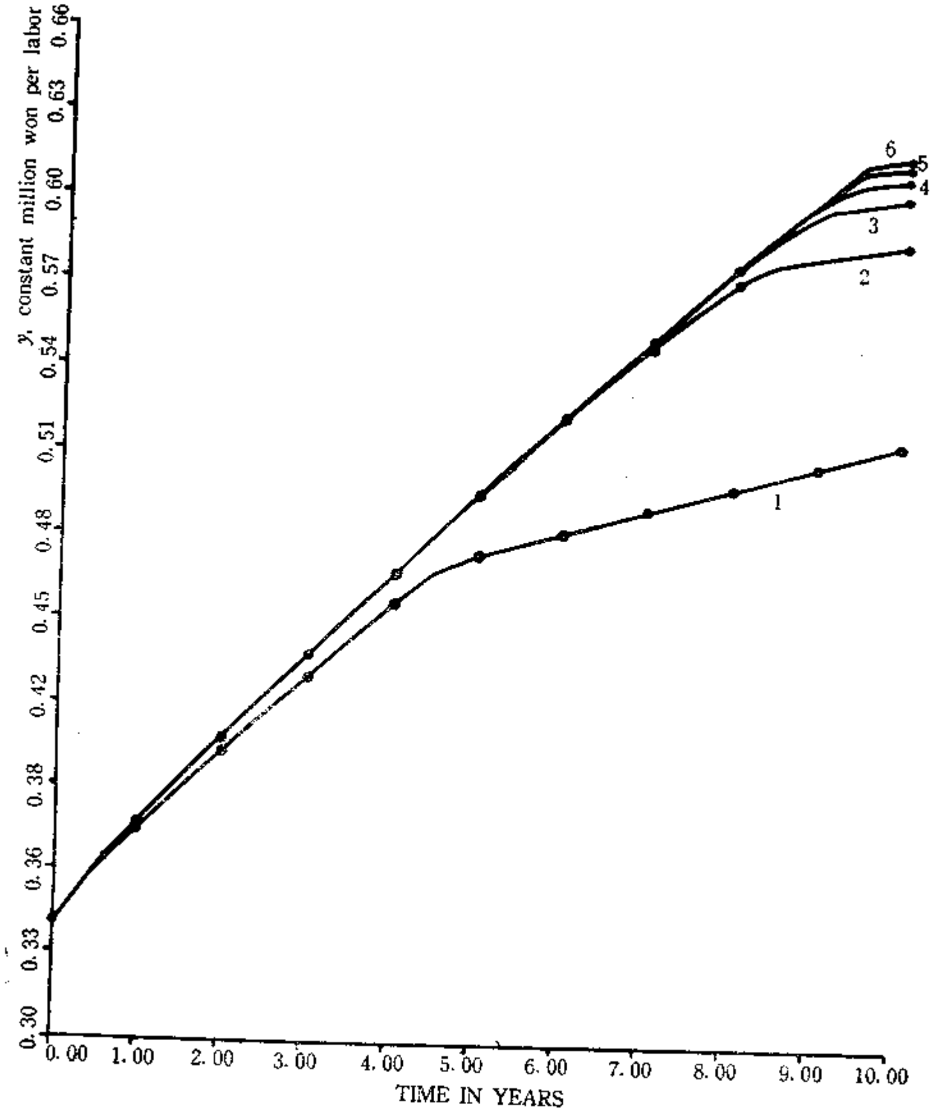

Figure 7. Trajectory of Output per Labor, $y(t)$