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ON THE CONFORMAL AREAL SPACES OF THE SUBMETRIC CLASS IV. THE CONFORMAL CONNECTION PARAMETERS

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The present paper is a further outcome of our previous papers [1, 2, 3][1] on the studies of Areal spaces of the submetric class in connection with the conformal correspondence. The prime purpose of this paper is to deduce the conformal connection parameters in the Areal space of the submetric class with respect to the Davies' connection parameters [6]. The notations used in this paper are the same as those employed in our earlier paper [1] and in [6] without explanations.

Let us consider two distinct n-dimensional Areal spaces $A_n^{(m)}$ and $\overline{A}_n^{(m)}$ of the submetric class, the fundamental functions of which are given by $F(x^i, p^i_\alpha)$ and $\overline{F}(x^i, p^i_\alpha)$ whith the same system of coordinates. These two spaces $A_n^{(m)}$ and $\overline{A}_n^{(m)}$ are said to be conformally related if their respective four-index metric tensors $g^{\alpha\beta}_{ij}$ and $\bar{g}^{\alpha\beta}_{ij}$ are connected by the relation

$$\bar{g}^{\alpha\beta}_{ij} = e^{2\sigma}\, g^{\alpha\beta}_{ij}, \tag{1}$$

where $e^{2\sigma}$ is a factor of proportionality and σ is at most a point function (theorem 3.1 [1]).

When two spaces $A_n^{(m)}$ and $\overline{A}_n^{(m)}$ are in the conformal correspondence, then with the aid of (1), it is easy to see that the normalized metric tensors (A. Kawaguchi and K. Tandai [4]) g_{ij} and $\bar{g}_{ij}$ of the spaces $A_n^{(m)}$ and $\overline{A}_n^{(m)}$ respectively are related [1] by

$$\bar{g}_{ij} = e^{2\sigma}\, g_{ij}, \tag{2}$$

and therefore

$$\bar{g}^{ij} = e^{-2\sigma}\, g^{ij}, \tag{3}$$

1) Numbers in brackets refer to the references at the end of this paper.
2) Latin indices $h, i, j, k, \cdots\cdots\cdots$ run from 1 to n and the Greek indices $\alpha, \beta, \gamma, \delta, \cdots\cdots$ from 1 to m throughout this paper.

where g^{ij} and $\bar{g}^{ij}$ are the contravariant counter-parts of the tensors g_{ij} and $\bar{g}_{ij}$ respectively.

Due to E. T. Davies [6], we also have a set of connection parameters $\mathring{\Gamma}^{h}_{ij}(x, p)$ in Areal spaces of the submetric class which are explicitly given by

(4) $$\mathring{\Gamma}^{h}_{ij}=g^{hk}\mathring{\Gamma}_{ijk}=\frac{1}{2}g^{hk}(e_i g_{jk}+e_j g_{ik}-e_k g_{ij}),$$

where

(5) $$e_i\equiv\partial_i-G^{\gamma}_{\alpha i}\,\partial^{\alpha}_{\gamma},$$

and $G^{\gamma}_{\alpha i}$ the so called G functions by E. T. Davies' are the well defined quantities in the space $A^{(m)}_{n}$, which have been determined in terms of the four-index metric tensor $g^{\alpha\beta}_{ij}$.

If we introduce the notations

$$\partial_k\equiv\frac{\partial}{\partial x^k}=,k \text{ and } \partial^{\alpha}_{\gamma}\equiv\frac{\partial}{\partial p^{\gamma}_{\alpha}}=;\,{}^{\alpha}_{\gamma},$$

then, the simplification of (4) by means of (5) yields

(6) $$\mathring{\Gamma}^{h}_{ij}=\frac{1}{2}g^{hk}[(g_{jk,i}+g_{ik,j}-g_{ij,k})-(G^{\gamma}_{\alpha i}g_{jk};{}^{\alpha}_{\gamma}+G^{\gamma}_{\alpha j}g_{ik};{}^{\alpha}_{\gamma}-G^{\gamma}_{\alpha k}g_{ij};{}^{\alpha}_{\gamma})].$$

Further, on making the use of the following notations

(7) $$\gamma^{h}_{ij}=g^{hk}\gamma_{ijk} \quad \text{with } \gamma_{ijk}=\frac{1}{2}(g_{jk,i}+g_{ik,j}-g_{ij,k}),$$

we can express (6) as

(8) $$\mathring{\Gamma}^{h}_{ij}=\gamma^{h}_{ij}-\frac{1}{2}g^{hk}(G^{\gamma}_{\alpha i}g_{jk};{}^{\alpha}_{\gamma}+G^{\gamma}_{\alpha j}g_{ik};{}^{\alpha}_{\gamma}-G^{\gamma}_{\alpha k}g_{ij};{}^{\alpha}_{\gamma}).$$

Now, in virtue of the relation ([5] p. 74, relation 5.7)

$$2C_{ij,}{}^{\alpha}_{k}=g_{ij};{}^{\alpha}_{k}+g_{mn};{}^{\alpha}_{k}\gamma^{m}_{i}\beta^{n}_{j}-g_{mn};{}^{\alpha}_{k}\gamma^{m}_{j}\beta^{n}_{i}.$$

and as a result of the symmetry properties of g_{ij}.
we have

$$g_{ij};{}^{\alpha}_{k}=C_{ij,}{}^{\alpha}_{k}+C_{ji,}{}^{\alpha}_{k}.$$

After this, following our previous paper [2], if we put $\underset{+}{C}_{ij,}{}^{\alpha}_{k}=C_{ij,}{}^{\alpha}_{k}+C_{ji,}{}^{\alpha}_{k}$, it is immediately obvious that

(9) $$g_{ij};{}^{\alpha}_{k}=\underset{+}{C}_{ij,}{}^{\alpha}_{k}.$$

Taking (9) as granted and on putting $g^{hj}\underset{+}{C}_{ij,}{}^{\alpha}_{k}=\underset{+}{C}^{h}_{i,}{}^{\alpha}_{k,}$ the definition (8) can finally be rewritten into the form

$$(10)\quad \overset{\circ}{\Gamma}{}^h_{ij} = \gamma^h_{ij} - \frac{1}{2}(G^{\gamma}_{\alpha i} \underset{+}{C}{}^{h\ \alpha}_{j,\ \gamma} + G^{\gamma}_{\alpha j} \underset{+}{C}{}^{h\ \alpha}_{i,\ \gamma} - g^{hk}\ G^{\gamma}_{\alpha k} \underset{+}{C}{}_{ij,\ \gamma}^{\ \ \alpha}).$$

Henceforth, we shall denote the quantities referring to $\overline{A}{}^{(m)}_n$ by bar overhead throughout this paper.

Let $\bar{\Gamma}{}^h_{ij}$ be the corresponding connection parameters of the space $\overline{A}{}^{(m)}_n$ then, with the understanding of the above relation, these are given by

$$(11)\quad \bar{\Gamma}{}^h_{ij} = \bar{\gamma}{}^h_{ij} - \frac{1}{2}(\overline{G}{}^{\gamma}_{\alpha i} \underset{+}{\overline{C}}{}^{h\ \alpha}_{j,\ \gamma} + \overline{G}{}^{\gamma}_{\alpha j} \underset{+}{\overline{C}}{}^{h\ \alpha}_{i,\ \gamma} - \bar{g}^{hk} \overline{G}{}^{\gamma}_{\alpha k} \underset{+}{\overline{C}}{}_{ij,\ \gamma}^{\ \ \alpha}).$$

Since σ is at most a point function, therefore (2) gives us

$$(12)\qquad \bar{g}_{ij,\,k} = e^{2\sigma}(2\sigma_k\ g_{ij} + g_{ij,\,k}),\ \text{where } \sigma_k \equiv \sigma,_{k},$$

from which it follows that

$$(13)\qquad \bar{\gamma}{}^h_{ij} = \gamma^h_{ij} + \sigma_j \delta^h_i + \sigma_i \delta^h_j - g^{hk} g_{ij} \sigma_k,$$

where we have made the use of the relation (7). Contracting (13) with respect to $h=i$, we obtain

$$(14)\qquad \sigma_j = \frac{1}{n}(\bar{\gamma}{}^h_{hj} - \gamma^h_{hj}).$$

Again contracting (10) and (11) with regard to $h=i$ and on subtracting the former relation from the latter one, if we employ (14) in the so obtained result, we find that

$$(15)\quad \sigma_j = \frac{1}{2n}\Big[\,(2\bar{\Gamma}{}^h_{hj} + \overline{G}{}^{\gamma}_{\alpha h} \underset{+}{\overline{C}}{}^{h\ \alpha}_{j,\ \gamma}) - (2\overset{\circ}{\Gamma}{}^h_{hj} + G^{\gamma}_{\alpha h} \underset{+}{C}{}^{h\ \alpha}_{j,\ \gamma})\,\Big].$$

Subtracting the relation (10) from (11) and on making the use of results (13) and (15), we get finally

$$(16)\quad \left\{ \begin{aligned} & \bar{\Gamma}{}^h_{ij} - \frac{1}{2n}\Big[(2\bar{\Gamma}{}^l_{lj} + \overline{G}{}^{\gamma}_{\alpha l} \underset{+}{\overline{C}}{}^{l\ \alpha}_{j,\ \gamma})\ \delta^h_i + (2\bar{\Gamma}{}^l_{li} + \overline{G}{}^{\gamma}_{\alpha l} \underset{+}{\overline{C}}{}^{l\ \alpha}_{i,\ \gamma})\ \delta^h_j \\ & \qquad - \bar{g}^{hk}\ \bar{g}_{ij}(2\bar{\Gamma}{}^l_{lk} + \overline{G}{}^{\gamma}_{\alpha l} \underset{+}{\overline{C}}{}^{l\ \alpha}_{k,\ \gamma})\ \Big] \\ & \quad + \frac{1}{2}(\overline{G}{}^{\gamma}_{\alpha i} \underset{+}{\overline{C}}{}^{h\ \alpha}_{j,\ \gamma} + \overline{G}{}^{\gamma}_{\alpha j} \underset{+}{\overline{C}}{}^{h\ \alpha}_{i,\ \gamma} - \bar{g}^{hk} \overline{G}{}^{\gamma}_{\alpha k} \underset{+}{\overline{C}}{}_{ij,\ \gamma}^{\ \ \alpha}) \\ & = \overset{\circ}{\Gamma}{}^h_{ij} - \frac{1}{2n}\Big[(2\overset{\circ}{\Gamma}{}^l_{lj} + G^{\gamma}_{\alpha l} \underset{+}{C}{}^{l\ \alpha}_{j,\ \gamma})\ \delta^h_i + (2\overset{\circ}{\Gamma}{}^l_{li} + G^{\gamma}_{\alpha l} \underset{+}{C}{}^{l\ \alpha}_{i,\ \gamma})\ \delta^h_j \\ & \quad - g^{hk} g_{ij}(2\overset{\circ}{\Gamma}{}^l_{lk} + G^{\gamma}_{\alpha l} \underset{+}{C}{}^{l\ \alpha}_{k,\ \gamma})\ \Big] \\ & \quad + \frac{1}{2}(G^{\gamma}_{\alpha i} \underset{+}{C}{}^{h\ \alpha}_{j,\ \gamma} + G^{\gamma}_{\alpha j} \underset{+}{C}{}^{h\ \alpha}_{i,\ \gamma} - g^{hk} G^{\gamma}_{\alpha k} \underset{+}{C}{}_{ij,\ \gamma}^{\ \ \alpha}), \end{aligned} \right.$$

where we have also taken the help of the relations (2) and (3).

Now, if we put

$$(17)\begin{cases} \Pi_{ij}^{h}(x,p)=\mathring{\Gamma}_{ij}^{h}-\frac{1}{2n}\Big[(2\mathring{\Gamma}_{lj}^{l}+G_{\alpha l}^{\gamma}\underset{+}{C}_{j,\gamma}^{l\alpha})\delta_i^h+(2\mathring{\Gamma}_{li}^{l}+G_{\alpha l}^{\gamma}\underset{+}{C}_{i,\gamma}^{l\alpha})\delta_j^h \\ \qquad -g^{hk}g_{ij}(2\mathring{\Gamma}_{lk}^{l}+G_{\alpha l}^{\gamma}\underset{+}{C}_{k,\gamma}^{l\alpha})\Big] \\ \qquad +\frac{1}{2}(G_{\alpha i}^{\gamma}\underset{+}{C}_{j,\gamma}^{h\alpha}+G_{\alpha j}^{\gamma}\underset{+}{C}_{i,\gamma}^{h\alpha}-g^{hk}G_{\alpha k}^{\gamma}\underset{+}{C}_{ij,\gamma}^{\alpha}), \end{cases}$$

the relation (16) reduces to

$$\overline{\Pi}_{ij}^{h}(x,p)=\Pi_{ij}^{h}(x,p),$$

which shows that the quantities $\Pi_{ij}^{h}(x,p)$ defined by (17) are invariant under the conformal transformation (1).

We now remember that if the space $A_n^{(m)}$ under consideration is a Riemannian spaces in particular, $C_{ij,k}^{\alpha}=0$ (Theorem 1, due to K. Tandai [7]). Consequently, we have $\underset{+}{C}_{ij,k}^{\alpha}=0$. In such a case the relation (10) becomes

$$\mathring{\Gamma}_{ij}^{h}=\gamma_{ij}^{h}\equiv\left\{ {h \atop i\,j} \right\},$$

and (17) reduces to

$$\Pi_{ij}^{h}(x)=\left\{ {h \atop i\,j} \right\}-\frac{1}{n}\left[\left\{ {l \atop l\,j} \right\}\delta_i^h+\left\{ {l \atop l\,i} \right\}\delta_j^h-g^{hk}g_{ij}\left\{ {l \atop l\,k} \right\}\right],$$

which are the quantities derived by J. M. Thomas[3], known as the conofrmal connection parameters of the Riemannian spaces. We shall call the quantities $\Pi_{ij}^{h}(x,p)$ 'the conformal connection parameters' of the Areal space of the submetric class.

From (17), by means of the straightforward calculation, we can see with ease that the quantities $\Pi_{ij}^{h}(x,p)$ satisfy the following relations:

$$\Pi_{ij}^{h}(x,p)=\Pi_{ji}^{h}(x,p),\quad \Pi_{hj}^{h}(x,p)=0,\quad \Pi_{ij;\ \gamma}^{h\ \ \lambda}\,p_{\mu}^{\gamma}=0.$$

Furthermore, from (14) and (15), we notice that the quantities $\theta_j(x,p)$ defined by

$$\theta_j(x,p)=2\mathring{\Gamma}_{hj}^{h}-2\gamma_{hj}^{h}+G_{\alpha h}^{\gamma}\underset{+}{C}_{j,\gamma}^{h\alpha},$$

are also invariant, i.e, $\overline{\theta}_j(x,p)=\theta_j(x,p)$, under the conformal transformation (1).

This completes our discussion.

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3) See Eisenhart, L. P. [8] p. 94; Thomas, J. M.: Conformal Correspondence of Riemannian Spaces. Proc. of the Nat. Acad. of Sciences, Vol. 11(1925), p. 257.

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REFFRENCES

[1] Om P. Singh, *On the conformal Areal spaces of the submetric class. The transformation of curvature tensors*, Math. Nachr., 58(1973), 145—159.

[2] Om P. Singh, *On the conformal Areal spaces of the submetric class* II. *The deductions of some identities*, Math Nachr., 58(1973), 161—168.

[3] Om P. Singh, *On the conformal Areal spaces of the submetric class* III. *The Bianchi identities.* (to be appeared).

[4] A. Kawaguchi, and K. Tandai, *On Areal spaces V. Normalized metric tensor and connection parameters in a space of the submetric class*, Tensor N. S. 2(1952), 47—58.

[5] E. T. Davies, *Areal spaces*, Annali di Mat. (IV), 55(1961), 63—76.

[6] E. T. Davies, *A Geometrical theory of multiple intgeral problems in the calculus of variations*, Aequationes Mathematicae, 7(1972)173—181.

[7] K. Tandai, *On Areal spaces* VI. *On the characterization of metric spaces*, Tensor, N. S., 3(1953), 40—45.

[8] L. P. Eisenhart, *Riemanian Geometry*, Princeton, (1926).