Kyungpook Math. J.
Volume 18, Number 2
December, 1978

ON A DIFFERENTIABLE MANIFOLD WITH $f(3,-1)$-STRUCTURE OF RANK r, (II)

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1. Introduction

Let us consider an n-dimensional C^{∞} differentiable manifold and a tensor field f of type (1, 1) such that

$$f_i^h f_h^k f_k^j - f_i^j = 0, \quad f \neq 0, 1; \tag{1.1}$$

where the indices $i, j, k, \cdots, h, \cdots$ vary from 1 to n. Let L and M be the complementary distributions corresponding to the projection tensors l_h^i and m_h^i respectively, where

$$\text{a) } l_h^i \stackrel{\text{def}}{=} f_r^i f_h^r, \quad \text{b) } m_h \stackrel{\text{def}}{=} l_h^i - f_r^i f_h^r. \tag{1.2}$$

If the rank of the structure tensor f is r, then such a structure is called an $f(3,-1)$-structure of rank r [1]. Also a manifold with an $f(3,-1)$-structure of rank r always admits a positive definite Riemannian metric g, such that

$$f_j^r f_i^s g_{rs} \stackrel{\text{def}}{=} g_{ji} - m_{ji}, \tag{1.3}$$

where $\qquad m_{ji} = m_j^r g_{ri}$.

Thus if an $f(3,-1)$-structure of rank r admits a positive definite Riemannian metric defined by (1, 3), then such a structure will be called $(f_r(r,-1), g)$-structure. If this $(f_r(3,-1), g)$-structure also satisfies the following relations:

$$N_{ji}^{\ \ h} \stackrel{\text{def}}{=} f_j^l \nabla_l f_i^h + f_i^l \nabla_l f_j^h - (\nabla_j f_i^l - \nabla_i f_j^l) f_l^h = 0,$$

$$F_{jih} \stackrel{\text{def}}{=} \nabla_j F_{ih} + \nabla_i F_{hj} + \nabla_h F_{ji} = 0,$$

$$M_{jih} \stackrel{\text{def}}{=} \nabla_j m_{ih} + \nabla_i m_{hj} + \nabla_h m_{ji} = 0,$$

then we call manifold M^n as a pseudo-manifold with $(f_r(3,-1), g)$-structure.

A vector field v^i in a pseudo manifold with $(f_r(3,-1), g)$-structure is called to be invariant with respect to the structure tensor f_j^i, that is f_j^i-invariant if the Lie-derivative of the structure tensor w.r.t. the vector field v^i is zero [2].

2. Infinitesimal projective transformation

Now we consider infinitesimal projective transformation with respect to f_j^i-invariant vector field v^i. We have [4],

$$\mathcal{L}\left\{ {i \atop j\ k} \right\} = \delta_j^i \phi_k + \delta_k^i \phi_j, \tag{2.1}$$

therefore

$$\mathcal{L}_v R_{lkj}{}^i = \delta_p^i \nabla_l \phi_j - \delta_l^i \nabla_k \phi_j, \tag{2.2}$$

where $\mathcal{L}_v$ stands for the Lie-derivative w.r.t. the vector V^i. The scalar ϕ is defined as $\nabla_\nu V^\nu / n+1$ and ϕ_i is grad. ϕ. Now we apply Ricci identity to the structure f_j^i. We have [5],

$$\nabla_p \nabla_j f_i^h - \nabla_j \nabla_p f_i^h = f_i^l R_{pjl}{}^h - f_l^h R_{pji}{}^l. \tag{2.3}$$

THEOREM 2.1. *In a compact pseudo-manifold with $(f_r(3, -1), g)$-structure, an infinitesimal projective transformation with respect to an f_j^i-invariant vector field is necessarily an isometry.*

PROOF. Now taking Lie-derivative of (2.3) w.r.t. V^i, we get

$$f_i^l \mathcal{L}_v R_{pjl}{}^h - f_l^h \mathcal{L}_v R_{pji}{}^l = 0, \tag{2.4}$$

because for a pseudo manifold with $(f_r(3, -1), g)$-structure

$$\nabla_p f_j^i = 0. \text{ Also } \mathcal{L}_v f_j^i = 0$$

Now by virtue of (2.2) and (2.4) we obtain

$$f_i^l (\delta_j^h \nabla_p \phi_l - \delta_p^h \nabla_j \phi_l) - (f_j^h \nabla_p \phi_i - f_p^h \nabla_j \phi_i) = 0. \tag{2.5}$$

Transvecting(2.5) with g^{ji} and multiplying the resulting equation by f_h^p, we get

$$l^{pl} \nabla_p \phi_l = -\frac{1}{2} l_i^i \nabla_a \phi^a, \tag{2.6}$$

in view of (1.2)a.

Applying Ricci identity to the projection tensor m_j^i, we have

$$\nabla_p \nabla_j m_i^h - \nabla_j \nabla_p m_i^h = m_i^l R_{pjl}{}^h - m_l^h R_{pji}{}^l. \tag{2.7}$$

For a pseudo-manifold, $\nabla_p f_j^i=0$ implies $\nabla_p l_j^i=0$; hence $\nabla_p m_j^i=0$. Also $\mathcal{L}_v f_p^i=0$ implies that $\mathcal{L}_v l_p^i=0$. Thus $\mathcal{L}_v m_p^i=0$. So the left hand side of (2.7) vanishes, hence we get

$$m_i^l R_{pjl}{}^h - m_l^h R_{pji}{}^l = 0. \tag{2.8}$$

Taking Lie-derivative of (2.8) w.r.t. V^i, we obtain

$$m_i^l \mathcal{L}_v R_{pjl}{}^h - m_l^h \mathcal{L}_v R_{pji}{}^l = 0. \tag{2.9}$$

By virtue of (2.2) and (2.9), we get

$$m_i^l(\delta_j^h \nabla_p \phi_l - \delta_p^h \nabla_j \phi_l) = m_j^h \nabla_p \phi_i - m_p^h \nabla_j \phi_i. \tag{2.10}$$

Multiplying (2.10) by g^{ji} and contracting h and p in the resulting equation we obtain

$$n \cdot m^{jl} \nabla_j \phi_l = m_i^i \nabla_a \phi^a. \tag{2.11}$$

Adding (2.6) and (2.11) we get

$$-2l^{lk} \nabla_k \phi_l + n \cdot m^{jl} \nabla_j \phi_l = n \nabla_a \phi^a \tag{2.12}$$

or

$$(n+2) m^{jl} \nabla_j \phi_l - 2\nabla_a \phi^a = n \nabla_a \phi^a. \tag{2.13}$$

This gives us that $m^{jl}\nabla_j\phi_l = \nabla_a \phi^a$. Hence from (2.11) and (2.13) we obtain

$$(1 - m_i^i/n) \nabla_a \phi^a = 0.$$

Since $m_i^i \neq n$, therefore we have

$$\nabla_a \phi^a = 0.$$

Hence if the space is compact and applying theorem of E. Hopf, we get $\phi=$ constant. Thus the transformation is necessarily an affine. Also in a compact manifold, an affine transformation is necessarily an isometry. Hence the theorem is proved.

3. Holomorphic projective transformation

Now we consider a Kählerian space W in a product manifold, that is, d-Kählerian, where d means decomposible, which is an enveloping space of the pseudo-manifold with $(f_r(3, -1), g)$-structure. We have [5]

$$\frac{d^2X^a}{d\mathscr{T}'^2} + \left\{ {a \atop c\ d} \right\} \frac{dX^c}{d\mathscr{T}'} \frac{dX^d}{d\mathscr{T}'} = \alpha' \frac{dX^a}{d\mathscr{T}'} + \beta' f_b^a \frac{dX^b}{d\mathscr{T}'},$$

where α' and β' are certain functions. We call such a curve a holomorphically planer curve and f_b^a is a product structure in W. a, b, c vary from 1 to the number which is the dimension of W. The H-plane curve in W gives rise to an H-plane curve in M^n which can be written as

$$\frac{d^2X^h}{d\mathscr{T}^2}+\left\{{h \atop i\ j}\right\}\frac{dX^j}{d\mathscr{T}}\frac{dX^i}{d\mathscr{T}}=\alpha_1\frac{dX^h}{d\mathscr{T}}+\beta_1 f_r{}^h\frac{dX^r}{d\mathscr{T}},$$

where f_r^h is $(f_r(3, -1), g)$-structure and α_1 and β_1 are suitably chosen functions. So holomorphically projective transformation can be defined in the pseudo-$(f_r(3, -1), g)$-manifold.

Now under this transformation, we have

$$\mathcal{L}_v\left\{{h \atop i\ j}\right\}=\delta_j^h\rho_i+\delta_i^h\rho_j-\tilde{\rho}_j f_i^h-\tilde{\rho}_i f_j^h, \tag{3.1}$$

where ρ_i is an arbitrary covariant vector field, and

$$\tilde{\rho}_i\overset{\text{def}}{=\!=}f_i^r\rho_r.$$

Consequently, we have

$$\mathcal{L}_v R_{kji}{}^h=\rho_{ki}\delta_j^h-\rho_{ji}\delta_k^h-f_i^r(\rho_{kr}f_j^h-\rho_{jr}f_k^h)-f_i^h(\rho_{kr}f_j^r-\rho_{jr}f_k^r), \tag{3.2}$$

where ρ_{ji} is a symmetric tensor defined by

$$\rho_{ji}=\nabla_j\rho_i-\nabla_i\rho_j+\tilde{\rho}_j\tilde{\rho}_i.$$

THEOREM 3.1. *If the f_j^i-invariant vector field in a pseudo-$(f_r(3, -1), g)$ manifold admits a holomorphically projective transformation, then*

$$\nabla^i\rho_i=m^{rl}\rho_r\rho_l.$$

PROOF. Let the holomorphically projective trasformation be taken with respect to f_j^i-invariant vector field. Putting the value of $\mathcal{L}_v R_{kjl}{}^h$ and $\mathcal{L}_v R_{kji}{}^l$ from (3.2) in (2.4) and then multiplying the resulting equation with g^{ji}, we get

$$l^{jr}\rho_{jr}-g^{jr}\rho_{jr}=0. \tag{3.3}$$

Now by virtue of (2.9) and (3.2) we get

$$m_i^l(\rho_{kl}\delta_j^h-\rho_{jl}\delta_k^h)-m_l^h(\rho_{ki}\delta_j^l-\rho_{ji}\delta_k^l)=0. \tag{3.4}$$

Contracting h and j in (3.4), we get

$$(n-1)m_i^l\rho_{kl}-m_l^l\rho_{ki}+m_k^j\rho_{ji}=0. \tag{3.5}$$

Transvecting (3.5) by g^{ik} and using (3.3), we obtain

$$m_l^l g^{ik}\rho_{ki}=0,$$

thus either $m_l^l=0$ or $g^{ik}\rho_{ki}=0$.

Let $m_l^l\neq 0$, therefore we have

$$g^{ik}\rho_{ki}=0$$

and hence

$$\nabla^i\rho_i=m^{rl}\rho_r\rho_l,$$

which proves the theorem.

ACKNOWLEDGEMENT. The second author is thankful to C. S. I. R., New Delhi for financial support by awarding Post Doctoral fellowship.

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