Kyungpook Math. J.
Volume 18, Number 2
December, 1978

ON MULTIPLICATION FORMULA FOR HYPERGEOMETRIC FUNCTIONS

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For Lauricella's hypergeometric function of n variables defined by ([5], p. 113)

$$F_A^{(n)}[a, b_1, \cdots, b_n ; c_1, \cdots, c_n ; x_1, \cdots, x_n]$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c_1)_{m_1} \cdots (c_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}$$

where $|x_1|+\cdots+|x_n|<1$ and $(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$, Srivastava ([6], p. 67) proved a multiplication formula

$$F_A^{(n)}[\sigma, -m_1, \cdots, -m_n ; \alpha_1+1, \cdots, \alpha_n+1 ; \lambda_1 x, \cdots, \lambda_n x]$$

$$= \sum_{k=0}^{m_1+\cdots+m_n} \binom{\alpha+k}{k} {}_2F_1\left(\begin{matrix}-k, \sigma \\ \alpha+1\end{matrix} ; x\right)$$

$$\times F_A^{(n+1)}[\alpha+1, -m_1, \cdots, -m_n, -k ; \alpha_1+1, \cdots, \alpha_n+1, \alpha+1 ; \lambda_1, \cdots, \lambda_n, 1] \qquad (1.1)$$

A special case of (1.1) for $n=1$ is

$$ {}_2F_1\left(\begin{matrix}-m, \sigma \\ \alpha+1\end{matrix} ; \lambda x\right) = \sum_{n=0}^{m} \binom{\beta+n}{n} {}_2F_1\left(\begin{matrix}-n, \sigma \\ \beta+1\end{matrix} ; x\right)$$

$$\times F_2[\beta+1, -m, -n ; \alpha+1, \beta+1 ; \lambda, 1] \qquad (1.2)$$

where F_2 is Appell's hypergeometric function ([3], p. 224).

The object of this paper is to generalize the result (1.2) in another form to obtain a multiplication formula

$$F\left[\begin{matrix}a, b : -m_2, \alpha_1+m_1+1 \\ c : \alpha_2+1, \ \alpha_1+1\end{matrix} ; \frac{y}{z}, \ -\frac{x}{z}\right]$$

$$= \sum_{k=0}^{m_1+m_2} \frac{(\alpha+1)_k}{k!} F_A^{(3)}[\alpha+1, -m_1, -m_2, -k ; \alpha_1+1, \alpha_2+1, \alpha+1 ; x, y, 1]$$

$$\times F\left[\begin{matrix}a, b : -, \alpha+k+1 \\ c : -, \alpha+1\end{matrix} ; \frac{x-1}{z}, \ -\frac{1}{z}\right] \qquad (1.3)$$

where

$$F\left[\begin{matrix}a, b : d, f \\ c : e, g\end{matrix} ; x, y\right] = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(a)_{r+s}(b)_{r+s}(d)_r(f)_s x^r y^s}{(c)_{r+s}(e)_r(g)_s \, r! s!}, \qquad (1.4)$$

$|x|<1$, $|y|<1$, is Kampe de Feriet's hypergeometric function of two variables (1). In Kampe de Feriet's notation, it is written as

$$F\left[\begin{array}{c|c|c} 2 & a,b & \\ 1 & d,f & x \\ 1 & c & y \\ 1 & e,g & \end{array}\right].$$

In order to prove (1.4), we consider a special case of the formula of Erdelyi ([2], p. 156)

$$L_{m_1}^{(\alpha_1)}(\lambda_1 x)\, L_{m_2}^{(\alpha_2)}(\lambda_2 x)=\sum_{k=0}^{m_1+m_2} c_2\, L_k^{(\alpha)}(x) \tag{1.5}$$

where

$$c_2=\binom{\alpha_1+m_1}{m_1}\binom{\alpha_2+m_2}{m_2}F_A^{(3)}\,[\alpha+1,-m_1,-m_2,-k\,;\,\alpha_1+1,\alpha_2+1,\alpha+1\,;\,\lambda_1,\lambda_2,1]$$

for $k\geqslant 0$ and $L_n^{(\alpha)}(x)$ is Laguerre polynomial. On multiplying both the sides of (1.5) by

$$e^{-\left(\lambda_1+\frac{1}{2}\beta\right)t}\, t^{\lambda}\, W_{\mu,\nu}(\beta t),$$

integrating with respect to t from 0 to ∞ by using a special case of the result ([4], p. 226)

$$\int_0^\infty e^{-\left(\lambda_1+\frac{1}{2}\beta\right)t}\, t^{\lambda}\, W_{\mu,\nu}(\beta t)\, L_{m_1}^{(\alpha_1)}(\lambda_1 t)\, L_{m_2}^{(\alpha_2)}(\lambda_2 t)\, dt$$

$$=\frac{\Gamma\left(\lambda+\nu+\frac{3}{2}\right)\Gamma\left(\lambda-\nu+\frac{3}{2}\right)(\alpha_1+1)_{m_1}(\alpha_2+1)_{m_2}}{m_1!\; m_2!\; \Gamma(\lambda-\mu+2)}\beta^{-\lambda-1}$$

$$F\left[\begin{matrix}\lambda+\nu+\frac{3}{2},\ \lambda-\nu+\frac{3}{2}\,:\,-m_2,\alpha_1+m_1+1 \\ \lambda-\mu+2\,;\,\alpha_2+1,\alpha_1+1\end{matrix}\,;\,\frac{\lambda_2}{\beta},\ -\frac{\lambda_1}{\beta}\right] \tag{1.6}$$

$$\mathrm{Re}(\beta)>\mathrm{Re}(\lambda_2)>0,\ \mathrm{Re}(\lambda_1)>0,\ \mathrm{Re}\left(\lambda+\nu+\frac{3}{2}\right)>0,$$

and replacing $\lambda+\frac{3}{2}+\nu$, $\lambda+\frac{3}{2}-\nu$, $\lambda-\mu+2$, λ_2, β and λ_1 by a, b, c, y, z and x respectively, we obtain (1.3).

When $m_2\to 0$, (1.3) reduces to

$$ {}_3F_2\left[\begin{matrix}a,b,\alpha_1+m_1+1 \\ c,\alpha_1+1\end{matrix}\,;\,-\frac{x}{z}\right]=\sum_{k=0}^{m_1}\frac{(\alpha+1)_k}{k!}$$

$$F_2[\alpha+1,-m_1,-k,\alpha_1+1,\alpha+1\,;\,x,1]\ F\left[\begin{matrix}a,b\,:\,\text{——},\alpha+k+1 \\ c\,:\,\text{——},\alpha+1\end{matrix}\,;\,\frac{x-1}{z},\ -\frac{1}{z}\right] \tag{1.7}$$

For $x=1$, (1.3) yields

$$F\left[\begin{matrix} a, b: -m_2, \alpha_1+m_1+1 \\ c: \alpha_2+1, \alpha_1+1 \end{matrix}; \frac{y}{z}, -\frac{1}{z}\right]$$

$$=\sum_{k=0}^{m_1+m_2} \frac{(\alpha+1)_k}{k!} F_A^{(3)}[\alpha+1, -m_1, -m_2, -k; \alpha_1+1, \alpha_2+1, \alpha+1; 1, y, 1]$$

$$\times {}_3F_2\left[\begin{matrix} a, b, \alpha+k+1 \\ c, \alpha+1 \end{matrix}; -\frac{1}{z}\right] \tag{1.8}$$

Setting $a=c$ in (1.3), we obtain a multiplication formula for hypergeometric functions in the form

$$F_2\left[b, -m_2, \alpha_1+m_1+1, \alpha_2+1, \alpha_1+1; \frac{y}{z}, -\frac{x}{z}\right]$$

$$=\sum_{k=0}^{m_1+m_2} \frac{(\alpha+1)_k}{k!} \left(\frac{z-x+1}{z}\right)^{-b} {}_2F_1\left(\begin{matrix} b, \alpha+k+1 \\ \alpha+1 \end{matrix}; -\frac{1}{z-x+1}\right)$$

$$\times F_A^{(3)}[\alpha+1, -m_1, -m_2, -k; \alpha_1+1, \alpha_2+1, \alpha+1; x, y, 1] \tag{1.9}$$

Finally, we let $m_1 \to 0$, $b=c$, $z=\frac{1}{x}$ and the result (1.8) gives

$$F_2[a, -m_2, \alpha_1+1; \alpha_2+1, \alpha_1+1; yx, -x]$$

$$=\sum_{k=0}^{m_2} \frac{(\alpha+1)_k}{k!} {}_2F_1\left(\begin{matrix} a, \alpha+k+1 \\ \alpha+1 \end{matrix}; -x\right) F_2[\alpha+1, -m_2, -k; \alpha_2+1, \alpha+1; y, 1] \tag{1,10}$$

which on applying ([3] p. 105(3) and p. 238(3)) and adjusting the parameters leads at once to the result (1.2) of Srivastava.

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