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ON F-STRUCTURE MANIFOLD

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Summary

The first part of this paper is devoted to the study of F-structure satisfying: $F^K+(-)^{K+1}F=0$ and $F^W+(-)^{W+1}F\neq 0$, for $1<W<K$. The case when K is odd and $K(\geqq 3)$ has been considered. In the later part some structures involving Lie-derivatives, exterior and co-derivatives have been studied.

1. Introduction

Let F be a non zero tensor field of type (1,1) and of class C^∞ on M^n such that [(2)]

(1.1) $F^K+(-)^{K+1}F=0$ and $F^W+(-)^{W+1}F\neq 0$ for $1<W<K$, where $K>2$.

Such a structure on M^n is called an *F-structure of rank 'r' and degree K*. If the rank of F is constant and $r=r(F)$, then M^n is called an *F-manifold of degree* K $(\geqq 3)$.

Let the operators on M^n be defined as follows [(2)]

(1.2) $\pi=(-)^K F^{K-1}$ and $m=I+(-)^{K+1}F^{K-1}$, where I is the identity operator on M^n.

Now we state the following theorems

THEOREM 1.1. *From the operators defined by* (1.2) *we have* $\pi+m=I$, $\pi^2=\pi$ *and* $m^2=m$. *For* F *satisfying* (1.1), *there exist complementary distributions* L *and* M *corresponding to the projection operators* π *and* m *respectively. If the rank of* F *is constant and is equal to* r *everywhere on n-dimensional manifold* M^n *then* $\dim L=r$ *and* $\dim M=(n-r)$.

THEOREM 1.2. *We have* (a) $F\pi=\pi F$, *and* $Fm=mF=0$, (b) $F^{K-1}\pi=-\pi$ *and* $F^{K-1}m=0$.

Thus $F^{\frac{K-1}{2}}$ *acts on* L *as an almost complex structure and on* M *as a null operator.*

PROOF. The proofs of theorems (1.1) and (1.2) are obvious in view of equations (1.1) and (1.2).

Let $\mathscr{F}(M)$ be the ring of real valued differentiable functions on M and $\mathscr{X}(M)$ be the module of derivatives of $\mathscr{F}(M)$. Then $\mathscr{X}(M)$ is Lie algebra over the real numbers and the elements of $\mathscr{X}(M)$ are called *vector fields.* Then M is equipped with (1, 1) tensor field F which is a linear map such that

$$F : \mathscr{X}(M) \longrightarrow \mathscr{X}(M).$$

Let M^n be of degree K and let K be a positive odd integer greater than 2 then we consider a positive definite Riemannian metric w.r.t. which L and M are orthogonal, so that

(1.3) $$g(X, Y) = g(HX, HY) + g(mX, Y), \text{ where } H = F^{\frac{K-1}{2}},$$
for all $X, Y \in \mathscr{X}(M)$.

Since the distributions L and M are orthogonal, using theorem (1.2) b, we obtain

(1.4) $$g(HX, Y) = g(H^2X, HY) \text{ and}$$

(1.5) $$g(X, HY) = g(HX, H^2Y) + g(mX, HY).$$

A 2- form has been defined as follows [(2)]

(1.6) $$H(X, Y) = -H(Y, X) = g(HX, Y).$$

In view of the definition of a Riemannian connection on M^n and Lie derivative L_X we have

(1.7) $$\text{a) } \nabla_X (F)(Y) = \nabla_X (FY) - F \nabla_X Y,$$
$$\text{b) } (L_X F)(Y) = [X, FY] - F[X, Y].$$

Making use of the theorem (1.2) a, and equations (1.7) a, b, we obtain

$$m(\nabla_X F)(mY) = 0, \quad \text{and } m(L_X F)(mY) = 0.$$

Since F^{K-1} is a (1, 1) tensor field we have

(1.8) $$\nabla_X(F^{K-1})(Y) = \nabla_X(F^{K-1}(Y)) - F^{K-1}\nabla_X Y.$$

The covariant and exterior derivative in M^n are defined as

(1.9) $$\nabla_X(H)(Y, Z) \overset{\text{def}}{=} g(\nabla_X(F)(Y), Z) \text{ and}$$

(1.10) $$dH(X, Y, Z) \overset{\text{def}}{=} (\nabla_X H)(Y, Z) + (\nabla_Y H)(Z, Y) + (\nabla_Z H)(X, Y).$$

In view of equation (1.7) a, we have the following identities,

(1.11) $$\nabla_X(H)(FY, F^{K-1}Z) = \nabla_X(H)(F^{K-1}Y, FZ),$$

(1.12) $$\nabla_X(H)(F^{K-1}Y, F^{K-1}Z) = -\nabla_X(H)(FY, FZ).$$

2. We have the following definitions [(1)]

(2.1) *F-Kählerian manifold* iff $\nabla_{FX} F \overset{\text{def}}{=} 0$.

(2.2) *F-almost Kählerian manifold* iff $dH(FX, FY, FZ) \overset{\text{def}}{=} 0$.

(2.3) *F-nearly Kählerian manifold* iff

$$\nabla_{FX}(F)(FY)+\nabla_{FY}(F)(FX) \overset{\text{def}}{=} 0$$

(2.4) *F-quasi Kählerian manifold* iff

$$\nabla_{FX}(F)(FY)+\nabla_{F}K-1_{X}(F)(F^{K-1}Y) \overset{\text{def}}{=} 0$$

(2.5) *F-Hermitian manifold* iff $N(FX, FY, FZ) \overset{\text{def}}{=} 0$,
for all $X, Y, Z \in \mathfrak{X}(M)$.

THEOREM 2.1. *The necessary and sufficient condition for an F-structure maniofold to be an F-nearly Kählerian manifold is that*

$$(2.6) \quad (-)^{K}F^{K-2}\{\nabla_{X}(F(FY))+\nabla_{FY}(F(FX))\}=\pi\{\nabla_{FX}(FY)+\nabla_{FY}(FX))\}.$$

PROOF. We have

$$(\nabla_{FX}F)(FY)=\nabla_{FY}(F(FY))-F\nabla_{FX}(FY). \text{ Thus,}$$

$$\nabla_{FX}(F(FY))+\nabla_{FY}(F(FX))=(\nabla_{FX}F)(FY)+F\nabla_{FX}(FY)$$
$$+(\nabla_{FY}F)(FX)+F\nabla_{FY}(FY).$$

In view of equation (2.3) we get

$$\nabla_{FX}(F(FY))+\nabla_{FY}(F(FX))=F\{\nabla_{FX}(FY)+\nabla_{FY}(FX)\}.$$

Operating above equation by $(-)^{K}F^{K-2}$ and on using (1.2) we obtain the result. The converse follows in and obvious manner.

THEOREM 2.2. *The necessary and sufficient condition for an F-structure manifold to be an F-quasi Kählerian manifold is that*

$$(2.7) \quad (-)^{K+1}F^{K-2}\{\nabla_{FX}(F(FY))+\nabla_{F}K-1_{X}(F(F^{K-1}Y)\}+$$
$$\pi\{\nabla_{FX}(FY)+\nabla_{F}K-1_{X}(F^{K-1}Y)\}=0.$$

PROOF. In consequence of (1.7) a, we get;

$$\nabla_{FX}(F(FY))=(\nabla_{FX}F)(FY)+F\nabla_{FX}(FY), \text{ and}$$

$$\nabla_{F}K-1_{X}(F(F^{K-1}Y))=(\nabla_{F}K-1_{X}F)\ (F^{K-1}Y)+F\nabla_{F}K-1_{X}(F^{K-1}Y).$$

Adding the last two equations and making use of (2.4) we get

$$\nabla_{FX}(F(FY))+\nabla_{F}K-1_{X}(F(F^{K-1}Y))=F\{\nabla_{FX}(FY)+\nabla_{F}K-1_{X}(F^{K-1}Y)\}.$$

Which on operating by $(-)^{K+1}\ F^{K-2}$ yields the result.

THEOREM 2.3. *The necessary and sufficient condition for an F-quasi Kählerian manifold to be F-Kählerian is that*

(2.8) $$\nabla_{FX}(FY)+\nabla_F K-1_X(F^{K-1}Y)=0$$

PROOF. By virtue of (2.4) we have $\nabla_{FX}F(FY)+\nabla_F K-1_X(F)(F^{K-1}Y))=0$ which in consequence of (1.7) a, yields

$$\nabla_{FX}(F(FY))-F\nabla_{FX}(FY)+\nabla_F K-1_X(F(F^{K-1}Y))$$
$$-F\nabla_F K-1_X(F^{K-1}Y)=0$$

which in view of definition (2.4) gives the required result.

THEOREM 2.4. *For an F-structure manifold, if any two of the following properties hold, the third is also satisfied,*

a) *it is F-nearly Kählerian,*

b) *it is F-quasi Kählerian, and*

c) *it is F-structure manifold for which*

(2.9) $$\nabla_{FY}F(FX)=\nabla_F K-1_X F(F^{K-1}Y).$$

PROOF. From definitions (2.3), (2.4), on substraction and making use of (2.9) we get the result.

THEOREM 2.5. *The necessary and sufficient condition for an F-structure manifold to be an F-almost Kählerian is that*

$$\nabla_F K-1_X(H)\ (F^{K-1}Y,FZ)+\nabla_F K-1_Y\ (H)\ (F^{K-1}Z,FX)$$
$$+\nabla_F K-1_Z(H)\ (F^{K-1}X,FY)=0.$$

PROOF. By virtue of equation (2.2) we obtain

$$\nabla_F K-1_X(H)\ (F^{K-1}Y,FZ)=-\nabla_F K-1_Y(H)(FZ,\ F^{K-1}X)$$
$$-\nabla_{FZ}(H)\ (F^{K-1}X,\ F^{K-1}Y),$$

(2.10) $$\nabla_F K-1_X(H)(F^{K-1}Y,FZ)=-\nabla_F K-1_Y(H)\ (FZ,\ F^{K-1}X)$$
$$+\nabla_{FZ}(H)\ (FX,\ FY).$$

Similarly we get

(2.11) $$\nabla_F K-1_Y(H)(F^{K-1}Z,FX)=-\nabla_F K-1_Z(H)\ (FX,\ F^{K-1}Y)$$
$$+\nabla_{FX}(H)\ (FY,FZ),$$ and

(2.12) $$\nabla_F K-1_Z(H)(F^{K-1}X, FY)=-\nabla_F K-1_X(H)\ (FY,\ F^{K-1}Z) + \nabla_{FY}\ (H)\ (FZ, FX).$$

Adding (2.10), (2,11), (2.12) and using (1.11) we get

$$2\{\nabla_F K-1_X(H)\ (F^{K-1}Y,\ FZ)+\nabla_F K-1_Y(H)\ (F^{K-1}Z, FX) +\nabla_F K-1_Z(H)(F^{K-1}X, FY)\}=dH(FX, FY, FZ)$$

which on using (2.2) yields

$$\nabla_F K-1_X(H)(F^{K-1}Y,\ FZ)+\nabla_F K-1_Y(H)\ (F^{K-1}Z, FX) +\nabla_F K-1_Z(H)\ (F^{K-1}X, FY)=0.$$

Gray [(1)] defined connection preserving structures for a Hermitian manifold. We shall now define an F-structure which is connection preserving as follows

(2.13) $$\nabla_F K-2_X\,(F^{K-2}Y)\overset{\text{def}}{=}\nabla_X Y$$

THEOREM 2.6. *If F-structure is connection preserving then F-quasi Kählerian manifold is F-Kählerian.*

PROOF. Let the manifold be F-quasi Kählerian then

$$\nabla_{FX}F(FY)=-\nabla_F K-1_X\,F(F^{K-1}Y)$$

which with the help of (1.7)a, gives

$$\nabla_{FX}\,F(FY)=-\{\nabla_F K-1_X\,F(F^{K-1}Y)-F\nabla_F K-1_X\,(F^{K-1}Y)\}.$$

which on arrangement of F^{K-1} etc. becomes

$$=-\{\nabla_F K-2_{(FX)}\ F^2(F^{K-2}Y)-F\nabla_F K-2_{(FX)}F(F^{K-2}Y)\}$$

This equation on using (2.1), (2.13) and (1.7) a, yields

$$\nabla_{FX}F(FY)=-\{\nabla_{FX}(F^2Y)-F\nabla_{FX}(FY)\}$$
$$=-\nabla_{FX}F(FY),\ \text{or}\ 2\nabla_{FX}F(FY)=0,$$

which in view of definition (2.1) shows that the manifold is F-Kählerian.

3. Operators $P(X, Y, Z)$ and $Q(X, Y, Z)$

Let us define the following operators

(3.1) $P(X, Y, Z)\overset{\text{def}}{=}\nabla_X(H)(Y,\ Z)+\nabla_Y(H)(X, Z)$ and

(3.2) $Q(X, Y, Z)\overset{\text{def}}{=}\nabla_X(H)\ (Y, Z)+\nabla_Y(H)(Z, X)+\nabla_Z(H)\ (X, Y).$

THEOREM 3.1. *For an F-almost Kählerian manifold we have*

(3.3) $$P(FX,\ F^{K-1}Y,\ F^{K-1}Z)+P(FY,\ F^{K-1}Z,\ F^{K-1}X)+P(FZ,\ F^{K-1}X,\ F^{K-1}Y)=0.$$

PROOF. In view of (3.1) we get

$$P(FX,\ F^{K-1}Y,\ F^{K-1}Z)=\nabla_{FX}(H)\ (F^{K-1}Y,\ F^{K-1}Z)+\nabla_{F^{K-1}Y}(H)(FX, F^{K-1}Z);\ \text{thus}$$

$$\begin{aligned}&P(FX, F^{K-1}Y,\ F^{K-1}Z)+P(FY,\ F^{K-1}Z,\ F^{K-1}X)+P(FZ,\ F^{K-1}X,\ F^{K-1}Y)\\&=-\{\nabla_{FX}(H)\ (FY,\ FZ)+\nabla_{FY}(H)\ (FZ, FX)+\nabla_{FZ}(H)\ (FX,\ FY)\}\\&=-\{\nabla_{F^{K-1}X}(H)\ (F^{K-1}Y,\ FZ)+\nabla_{F^{K-1}Y}(H)\ (F^{K-1}Z,\ FX)+\nabla_{F^{K-1}Z}(H)(F^{K-1}X, FY)\}.\end{aligned}$$

Now in consequence of the theorem (2.5) and the equation (2.2) we obtain theorem (3.1).

THEOREM 3.2. *If an F-structure manifold has the following two properties, i.e.*

a) *it is an F-almost Kählerian manifold,*

b) *it is an F-nearly Kählerian manifold then*

(3.4) $$\nabla_{F^{K-1}Z}(H)\ (FX,\ F^{K-1}Y)=2\nabla_{FX}(H)\ (FY,\ FZ).$$

PROOF. In view of equation (2.2) we have

(3.5) $$Q(FX, FY, FZ)=\nabla_{FX}(H)(FY, FZ)+\nabla_{FY}(H)\ (FZ, FX)+\nabla_{FZ}(H)\ (FX,\ FY)$$

(3.6) $$P(FX, FY, FZ)=\nabla_{FX}(H)\ (FY, FZ)+\nabla_{FY}(H)\ (FX, FZ)$$

Adding (3.5) and (3.6) we obtain

$$Q(FX, FY, FZ)+P(FX, FY, FZ)=2\nabla_{FX}(H)\ (FY, FZ)+\nabla_{FZ}(H)\ (FX, FY)$$

which for an F-almost Kählerian and F-nearly Kählerian manifold gives

$$2\nabla_{FX}(H)\ (FY, FZ)+\nabla_{FZ}(H)\ (FX, FY)=0.$$

$$\text{or, }2\nabla_{FX}(H)\ (FY, FZ)=-\nabla_{FZ}(H)\ (FX, FY)$$

$$\text{or, }-2\nabla_{FX}(H)\ (F^{K-1}Y,\ F^{K-1}Z)=\nabla_{F^{K-1}Z}(H)\ (FX, F^{K-1}Y)$$

$$\text{or, }2\nabla_{FX}(H)\ (FY,\ FZ)=\nabla_{F^{K-1}Z}(H)\ (FX,\ F^{K-1}Y).$$

COROLLARY. *For an F-nearly Kählerian manifold we have*

(3.7) $$\nabla_F K-1_X(H)\ (F^{K-1}Y,\ FZ)+\nabla_F K-1_Y(H)\ (F^{K-1}X, FZ)=0$$

PROOF. The proof follows immediately, making use of (1.11).

THEOREM 3.3. *For an F-nearly Kählerian manifold we have*

(3.8) $$Q(FX,\ F^{K-1}Y,\ F^{K-1}Z)+Q(FY, F^{K-1}X,\ F^{K-1}Z) =\nabla_F K-1_X(H)\ (F^{K-1}Y,\ FZ)\ +\nabla_F K-1_Y(H)\ (F^{K-1}Z,\ FX).$$

PROOF. In consequence of equation (3.2) we have

$$\begin{aligned} &Q(FX,\ F^{K-1}Y,\ F^{K-1}Z)+Q(FY, F^{K-1}X,\ F^{K-1}Z)\\ &\quad=-\{\nabla_{FX}(H)\ (FY,\ FZ)+\nabla_{FY}(H)\ (FX,\ FZ)\}\\ &\quad\quad+\nabla_F K-1_Y(H)\ (F^{K-1}Z,\ FX)+\nabla_F K-1_X(H)\ (FY,\ F^{K-1}Z)\\ &\quad\quad+\nabla_F K-1_Z(H)(FY, F^{K-1}X)+\nabla_F K-1_Z(H)\ (FX,\ F^{K-1}Y)\end{aligned}$$

which with the help of (1.11) and (2.3) give the result.

THEOREM 3.4. *For an F-nearly Kählerian manifold*

$$P(F^{K-1}X,\ F^{K-1}Y,\ FZ)+P(FX, F^{K-1}Y,\ F^{K-1}Z) +P(F^{K-1}X,\ FY,\ F^{K-1}Z)=0.$$

PROOF. The proof follows at once after making use of equations (3.1), (1.11), (1.12), and (3.7).

COROLLARY. *In an F-structure manifold following identities hold*

(3.9) $$Q(F^{K-1}X,\ F^{K-1}Y,\ FZ)=Q(FY, FX, FZ),$$

(3.10) $$Q(FX,\ F^{K-1}Y, FZ)=Q(FY, F^{K-1}X,\ FZ),$$

(3.11) $$Q(F^{K-1}X,\ F^{K-1}Y,\ F^{K-1}Z)=Q(FY,\ FX,\ F^{K-1}Z).$$

PROOF. The proof is obvious.

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