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ANTI-INVARIANT SUBMANIFOLDS OF REAL CODIMENSION OF A COMPLEX PROJECTIVE SPACE

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1. Introduction

As is well known, the unit hypersphere S^{2n+1} in an $(n+1)$-dimensional complex number space C^{n+1}, which will be identified naturally with $R^{2(n+1)}$, is a principal circle bundle over a complex projective space CP^n, and the Riemannian structure on CP^n is given by $\tilde{\pi}: S^{2n+1} \longrightarrow CP^n$ the natural projection of S^{2n+1} onto CP^n which is defined by the Hopf-fibration [6,7]. Thus the theory of submersion is one of the most useful tools for studying a complex projective space and its submanifold. In this point of view, H.B. Lawson [1], Y. Maeda [3] and M. Okumura [4] studied real hypersurfaces of a complex projective space.

On the other hand, K. Yano and M. Kon [9] proved

THEOREM A. *Let M be an $(m+1)$-dimensional compact orientable anti-invariant submanifold with parallel second fundamental form of S^{2n+1}. If the normal connection of M is flat, then*

$$M = S^1(r_1)\times\cdots\times S^1(r_{m+1}) \text{ in an } S^{2m+1} \text{ in } S^{2n+1},$$

where $r_1^2+\cdots+r_{m+1}^2=1$.

Using Theorem A, Okumura [6] have proved

THEOREM B. *Let M be a compact n-dimensional $(n>1)$ anti-invariant submanifold of a complex projective space CP^n with trivial normal connection. If the mean curvature vector field of M is parallel with respect to the normal connection and satisfies $H_BU_A=H_AU_B$ for $A,B=1,2,\cdots,n$, then $\pi^{-1}(M)$ is $S^1(r_1)\times\cdots\times S^1(r_{n+1})$, where $S^1(r_i)$ denotes the circle of radius r_i. Consequently M is diffeomorphic to n-product of circles.*

In this paper we also consider a submanifold M of CP^n which is a base space of a circle bundle $\overline{M}$ over M, where $\overline{M}$ is a submanifold of S^{2n+1}.

In 2, we state some fundamental formulas for submanifolds of Kaehlerian manifold and in 3, we recall fundamental equations of a submersion which are introduced by B. O'Neill [6], K. Yano and S. Ishihara [7]. Then, in 4 we consider a submanifold $\overline{M}$ of S^{2n+1} which is a circle bundle over a submanifold M of CP^n. Here we relate second fundamental tensor of the submanifolds $\overline{M}$ and M. The last section 5 is devoted to establish fundamental relations of the submersion $\tilde{\pi}: S^{2n+1}\longrightarrow CP^n$ and $\pi: \overline{M}\longrightarrow M$ in the case that the submanifold M is anti-invariant. And we find some necessary conditions for anti-invariant submanfold M with parallel second fundamental tensor to be a model subspace $S^1(r_1)\times\cdots\times S^1(r_{n+1})/\sim\ , r_2^1+\cdots+r_{n+1}^2=1$, appeared in Theorem B by using Theorem A. Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be differentiable and of class C^∞. We use in the present paper systems of indices as follows:

$$\kappa,\lambda,\mu,\nu=1,2,\cdots,2n+1\,;\ h,i,j,k=1,2,\cdots,2n,$$
$$\alpha,\beta,\gamma,\delta=1,2,\cdots,m+1\,;\ a,b,c,d,e=1,2,\cdots,m,$$
$$x,y,z,w=1,2,\cdots,2n-m.$$

The summation convention will be used with respect to those systems of indices.

2. Submanifolds of Kaehlerian manifolds

Let $\tilde{M}$ be a $2n$-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{\tilde{U}: y^j\}$ and denote by g_{ji} components of the Hermitian metric tensor and by ϕ_j^i those of the almost complex structure of $\tilde{M}$. Then we have

$$\phi_h^i\phi_j^h=-\delta_j^i, \tag{2.1}$$

$$\phi_j^h\phi_i^k g_{hk}=g_{ji}, \tag{2.2}$$

and, denoting by $\tilde{\nabla}_j$ the operator of covariant differentiation with respect to g_{ji},

$$\tilde{\nabla}_j\phi_i^h=0. \tag{2.3}$$

Let M be an m-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U: x^a\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i: M\longrightarrow\tilde{M}$. In the sequel we identify $i(M)$ with M itself and represent the immersion by

$$y^j=y^j(x^a). \tag{2.4}$$

We put

(2.5) $$B_b^j=\partial_b y^j,\quad \partial_b=\partial/\partial x^b$$

and denote by N_x^h mutually orthogonal unit normals to M. Then denoting by g_{cb} the fundamental metric tensor of M, we have

$$g_{cb}=B_c^j B_b^i g_{ji}$$

since the immersion is isometric. Therefore, denoting by ∇_b the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{cb}, we have equations of Gauss and Weingarten for M

(2.6) $$\nabla_c B_b^j=A_{cb}{}^x N_x^j,$$

(2.7) $$\nabla_c N_x^j=-A_{cx}^b B_b^j,$$

respectively, where $A_{cb}{}^x$ are the second fundamental tensors with respect to the normals N_x^j and $A_{cx}^b=A_{cax}g^{ab}=A_{ca}{}^y g^{ab} g_{xy}$, g_{xy} being the metric tensor of the normal bundle of M given by $g_{xy}=N_x^j N_y^i g_{ji}$ and $(g^{ba})=(g_{ba})^{-1}$.

Equations of Gauss, Codazzi and Ricci are respectively

(2.8) $$K_{dcb}{}^a=K_{kji}{}^h B_{dcbh}^{kjia}+A_{dx}^a A_{cb}{}^x-A_{cx}^a A_{db}{}^x,$$

(2.9) $$0=K_{kji}{}^h B_{dcb}^{kji} N_h^x-(\nabla_d A_{cb}{}^x-\nabla_c A_{db}{}^x),$$

and

(2.10) $$K_{dcy}{}^x=K_{kji}{}^h B_{dc}^{kj} N_y^i N_h^x+(A_{de}{}^x A_{cy}^e-A_{ce}{}^x A_{dy}^e),$$

where $B_{dcbh}^{kjia}=B_d^k B_c^j B_b^i B_h^a$, $B_{dcb}^{kji}=B_d^k B_c^j B_b^i$, $B_h^a=B_b^j g^{ba} g_{jh}$, $N_h^x=N_y^j g^{yx} g_{jh}$ and $K_{dcy}{}^x$ is the curvature tensor of the connection induced in the normal bundle.

We now consider the transforms $\phi_i^j B_b^i$ and $\phi_i^j N_x^i$ of B_b^i and N_x^i by the structure tensor ϕ_i^j. Then we can put in each coordinate neighborhood $U=\tilde{U}\cap M$

(2.11) $$\phi_i^j B_b^i=\phi_b^a B_a^j+\phi_b^x N_x^j,$$

(2.12) $$\phi_i^j N_x^i=-\phi_x^a B_a^j+\phi_x^y N_y^j$$

respectively.

Using $\phi_{ji}=-\phi_{ij}$, $\phi_{ji}=\phi_j^h g_{hi}$, we have, from (2.11) and (2.12),

(2.13) $$\phi_{bx}=\phi_{xb},$$

where $\phi_{bx}=\phi_b^y g_{yx}$ and $\phi_{xb}=\phi_x^a g_{ab}$ and

$$(2.14)\qquad \phi_{yx}=-\phi_{xy},$$

where $\phi_{yx}=\phi_y^z g_{zx}$.

Applying ϕ to (2.11) and (2.12) and using (2.1) and these equations, we can easily find

$$(2.15)\qquad \phi_a^b\phi_b^c+\delta_a^c=\phi_a^x\phi_x^c,$$

$$(2.16)\qquad \phi_a^b\phi_b^y+\phi_a^x\phi_x^y=0,\quad \phi_x^a\phi_a^b+\phi_x^y\phi_y^b=0,$$

$$(2.17)\qquad \phi_x^z\phi_z^y+\delta_x^y=\phi_x^a\phi_a^y.$$

Differentiating (2.11) and (2.12) covariantly along M and using (2.3) and the equations (2.6) and (2.7) of Gauss and Weingarten, we can verify that

$$(2.18)\qquad \nabla_b\phi_a^c=A_{bx}^c\phi_a^x-A_{ba}{}^x\,\phi_x^c,$$

$$(2.19)\qquad \nabla_b\phi_a^x=A_{ba}{}^y\,\phi_y^x-A_{bc}{}^x\,\phi_a^c,\quad \nabla_b\phi_x^a=A_{bx}^c\phi_c^a-A_{by}^a\phi_x^y,$$

$$(2.20)\qquad \nabla_b\phi_x^y=A_{ba}{}^y\,\phi_x^a-A_{bx}^a\quad\phi_a^y.$$

We now assume that the ambient manifold $\tilde{M}$ is of constant holomorphic sectional curvature c. Then it is well known that its curvature tensor $K_{kji}{}^h$ has the form

$$(2.21)\qquad K_{kji}{}^h=\frac{c}{4}(\delta_k^h g_{ji}-\delta_j^h g_{ki}+\phi_k^h\phi_{ji}-\phi_j^h\phi_{ki}-2\phi_{kj}\phi_i^h).$$

Therefore, substituting (2.21) into (2.8), (2.9) and (2.10), we can see that the equations of Gauss, Codazzi and Ricci are respectively given by

$$(2.22)\quad K_{dcb}{}^a=\frac{c}{4}(\delta_d^a g_{cb}-\delta_c^a g_{db}+\phi_d^a\phi_{cb}-\phi_c^a\phi_{db}-2\phi_{dc}\phi_b^a)+A_{dx}^a A_{cb}{}^x-A_{cx}^a A_{db}{}^x,$$

$$(2.23)\quad \nabla_d A_{cb}{}^x-\nabla_c A_{db}{}^x=\frac{c}{4}(\phi_d^x\phi_{cb}-\phi_c^x\phi_{db}-2\phi_{dc}\phi_b^x),$$

$$(2.24)\quad K_{dcy}{}^x=\frac{c}{4}(\phi_d^x\phi_{cy}-\phi_c^x\phi_{dy}-2\phi_{dc}\phi_y^x)+A_{de}{}^x A_{cy}^e-A_{ce}{}^x A_{dy}^e.$$

3. Submersion $\tilde{\pi}: S^{2n+1}\longrightarrow CP^n$ and immersion $i: M\longrightarrow CP^n$

Let $S^{2n+1}(1)$ be the hypersphere $\{(c^1,\cdots,c^{n+1})\,|\,|c^1|^2+\cdots+|c^{n+1}|^2=1\}$ of radius 1 in an $(n+1)$-dimensional space C^{n+1} of complexes, which will be identified naturally with $R^{2(n+1)}$. The sphere $S^{2n+1}(1)$ will be simply denoted by S^{2n+1}.

Let $\tilde{\pi}: S^{2n+1}\longrightarrow CP^n$ be the natural projection of S^{2n+1} onto a complex projective space CP^n which is defined by the Hopf fibration. We consider a Riemannian submersion $\pi : \overline{M} \longrightarrow M$ compatible with the Hopf fibration $\tilde{\pi} : S^{2n+1} \longrightarrow CP^n$, where M is a submanifold of codimension p in CP^n and $\overline{M} = \tilde{\pi}^{-1}(M)$ that of S^{2n+1}. More precisely speaking, $\pi : \overline{M} \longrightarrow M$ is a Riemannian submersion with totally geodesic fibres such that the following diagram is commutative:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\tilde{i}} & S^{2n+1} \\ \pi \downarrow & & \downarrow \tilde{\pi}, \\ M & \xrightarrow[i]{} & CP^n \end{array}$$

where $\tilde{i} : \overline{M} \longrightarrow S^{2n+1}$ and $i : M \longrightarrow CP^n$ are certain isometric immersions.

Covering S^{2n+1} by a system of coordinate neighborhoods $\{\hat{U} ; y^\kappa\}$ such that $\tilde{\pi}(\hat{U}) = \tilde{U}$ are coordinate neighborhoods of CP^n with local coordinate (y^j), we represent the projection $\tilde{\pi} : S^{2n+1} \longrightarrow CP^n$ by

(3.1) $$y^j = y^j(y^\kappa)$$

and put

(3.2) $$E^j_\kappa = \partial_\kappa y^j, \quad \partial_\kappa = \partial / \partial y^\kappa,$$

the rank of metric (E^j_κ) being always $2n$.

Let's denote by $\tilde{\xi}^\kappa$ components of $\tilde{\xi}$ the unit Sasakian structure vector in S^{2n+1}. Since the unit vector field $\tilde{\xi}$ is always tangent to the fibre $\tilde{\pi}^{-1}(\tilde{P})$, $\tilde{P} \in CP^n$ everywhere, E^j_κ and $\tilde{\xi}_\kappa$ form a local coframe in S^{2n+1}, where $\tilde{\xi}_\kappa = g_{\kappa\mu} \tilde{\xi}^\mu$ and $g_{\kappa\mu}$ denote the Riemannian metric tensor of S^{2n+1}. We denote by $\{E^\kappa_j, \tilde{\xi}^\kappa\}$ the frame corresponding to this coframe. We then have

(3.3) $$E^i_\kappa E^\kappa_j = \delta^i_j, \quad E^j_\kappa \tilde{\xi}^\kappa = 0, \quad \tilde{\xi}_\kappa E^\kappa_i = 0.$$

We now take coordinate neighborhoods $\{\overline{U} ; x^d\}$ of M such that $\pi(\overline{U}) = U$ are coordinate neighborhoods of M with local coordinates (x^a). Let the isometric immersions $\tilde{i}$ and i be locally expressed by $y^\kappa = y^\kappa(x^\alpha)$ and $y^j = y^j(x^a)$ in terms of local coordinates x^α in $\overline{U}(\subset \overline{M})$ and (x^a) in $U(\subset M)$ respectively. Then the commutativity $\tilde{\pi} \cdot \tilde{i} = i \cdot \pi$ of the diagram implies

$$y^j(x^a(x^\alpha)) = y^j(y^\kappa(x^\alpha)),$$

where we expressed the submersion π by $x^a = x^a(x^\alpha)$ locally, and hence

(3.4) $$B^j_a E^a_\alpha = E^j_\kappa B^\kappa_\alpha,$$

$B_a^j=\partial_a y^j$, $B_\alpha^\kappa=\partial_\alpha y^\kappa$ and $E_\alpha^a=\partial_\alpha x^a$.

For an arbitrary point $P\in M$ we choose unit normal vector fields N_x^j to M defined in a neighborhood U of P in such a way that $\{B_a^j, N_x^j\}$ span the tangent space of CP^n at $i(P)$. Let $\bar{P}$ be an arbitrary point of the fibre $\pi^{-1}(P)$ over P, then the lifts $N_x^\kappa=N_x^j E_j^\kappa$ of N_x^j are unit normal vector fields to $\overline{M}$ defined in the tubular neighborhood over U because of (3.4). Since $\tilde{\xi}^\kappa E_\kappa^j=0$, we can represent $\tilde{\xi}$ by

(3.5)
$$\tilde{\xi}^\kappa=\xi^\alpha B_\alpha^\kappa,$$

where ξ^α is a local vector field in $\overline{M}$. Using (3.4) and (3.5), we find

(3.6)
$$\xi_\alpha\xi^\alpha=1,\quad \xi^\alpha E_\alpha^a=0,$$

where $\xi_\alpha=\xi^\beta g_{\beta\alpha}$ and $g_{\beta\alpha}$ is the Riemannian metric tensor of $\overline{M}$ induced from that of S^{2n+1}. Therefore, $\{E_\alpha^a, \xi_\alpha\}$ is a local coframe in $\overline{M}$ corresponding to $\{E_\kappa^j, \tilde{\xi}_\kappa\}$ in S^{2n+1}. Denoting by $\{E_a^\alpha, \xi^\alpha\}$ the frame corresponding to this coframe, we have

(3.7)
$$E_\alpha^b E_a^\alpha=\delta_a^b,\quad \xi_\alpha E_b^\alpha=0,$$

and consequently

(3.8)
$$E_j^\kappa B_b^j=B_\alpha^\kappa E_b^\alpha$$

with the help of (3.4) and (3.6).

Denoting by $\left\{{\lambda \atop \mu\ \nu}\right\}$, $\left\{{i \atop j\ h}\right\}$, $\left\{{\alpha \atop \beta\ \gamma}\right\}$ and $\left\{{a \atop b\ c}\right\}$ the Christoffel symbols formed with the Riemannian metrics $g_{\mu\lambda}$, g_{ji}, $g_{\beta\alpha}$ and g_{ba} respectively, we put

$$D_\mu E_\lambda^i=\partial_\mu E_\lambda^i-\left\{{\kappa \atop \mu\ \lambda}\right\}E_\kappa^i+\left\{{i \atop j\ h}\right\}E_\mu^j E_\lambda^h,$$
$$D_\mu E_i^\lambda=\partial_\mu E_i^\lambda+\left\{{\lambda \atop \mu\ \kappa}\right\}E_i^\kappa-\left\{{h \atop j\ i}\right\}E_\mu^j E_h^\lambda,$$

and

$$\tilde{\nabla}_\beta E_\alpha^a=\partial_\beta E_\alpha^a-\left\{{\gamma \atop \beta\ \alpha}\right\}E_\gamma^a+\left\{{a \atop b\ c}\right\}E_\beta^b E_\alpha^c,$$
$$\tilde{\nabla}_\beta E_a^\alpha=\partial_\beta E_a^\alpha+\left\{{\alpha \atop \beta\ \gamma}\right\}E_a^\gamma-\left\{{c \atop b\ a}\right\}E_\beta^b E_c^\alpha.$$

Since the metrics $g_{\lambda\mu}$ and $g_{\alpha\beta}$ are invariant with respect to the submersions $\tilde{\pi}$ and π respectively, the van der Waerden-Bortolotti covariant derivatives of E_λ^i, E_i^λ and E_α^a, E_a^α are given by

(3.9) $$\begin{cases} D_\mu E^i_\lambda = h^i_j (E^i_\mu \tilde{\xi}_\lambda + \tilde{\xi}_\mu E^i_\lambda), \\ D_\mu E^\lambda_i = h_{ji} E^j_\mu \tilde{\xi}^\lambda - h^j_i \tilde{\xi}_\mu E^\lambda_j. \end{cases}$$

(3.10) $$\begin{cases} \tilde{\nabla}_\beta E^a_\alpha = h^a_b (E^b_\beta \xi_\alpha + \xi_\beta E^b_\alpha), \\ \tilde{\nabla}_\beta E^\alpha_a = h_{ba} E^b_\beta \xi_\alpha - h^b_a \xi_\beta E^\alpha_b \end{cases}$$

respectively, where $h^i_j = g^{ih} h_{ji}$, $h^a_b = g^{ac} h_{bc}$, h_{ji} being h_{ba} are the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively (See Ishihara and Konishi [2]).

On the other side the equations of Gauss and Weingarten for the immersion $\bar{i} : \overline{M} \longrightarrow S^{2n+1}$ are given by

(3.11) $$\begin{aligned} \tilde{\nabla}_\beta B^\kappa_\alpha &= \partial_\beta B^\kappa_\alpha + \left\{ {\kappa \atop \mu\ \lambda} \right\} B^\mu_\beta B^\lambda_\alpha - \left\{ {\gamma \atop \beta\ \alpha} \right\} B^\kappa_\gamma = A_{\beta\alpha}{}^x N^\kappa_x, \\ \tilde{\nabla}_\beta N^\kappa_x &= \partial_\beta N^\kappa_x + \left\{ {\kappa \atop \mu\ \lambda} \right\} B^\mu_\beta N^\lambda_x - \Gamma^y_{\beta x} N^\kappa_y = -A^\alpha_{\beta x} B^\kappa_\alpha, \end{aligned}$$

and those for the immersion $i : M \longrightarrow CP^n$ by

(3.12) $$\begin{aligned} \nabla_b B^i_a &= \partial_b B^i_a + \left\{ {i \atop j\ h} \right\} B^j_b B^h_a - \left\{ {c \atop b\ a} \right\} B^i_c = A_{ba}{}^x N^i_x, \\ \nabla_b N^i_x &= \partial_b N^i_x + \left\{ {i \atop j\ h} \right\} B^j_b N^h_x - \Gamma^y_{bx} N^j_y = -A^a_{bx} B^i_a, \end{aligned}$$

$\Gamma^y_{\beta x}$ and Γ^y_{bx} being components of the connections induced on the normal bundles $N(\overline{M})$ and $N(M)$ of $\overline{M}$ and M respectively, where $A^\alpha_{\beta x} = A_{\beta\gamma}{}^y g^{\alpha\gamma} g_{yx}$, $A_{\beta\alpha}{}^x$ and $A_{ba}{}^x$ are the second fundamental tensors of $\overline{M}$ and M with respect to the unit normals N^κ_x and N^j_x respectively. Moreover in such a case (3.4) and (3.8) imply

$$\nabla_b = E^\alpha_b \tilde{\nabla}_\alpha.$$

We now put $\phi^\lambda_\mu = D_\mu \tilde{\xi}^\lambda$. Then we have by definition of Sasakian structure

(3.13) $$\phi^\lambda_\mu \phi^\mu_\kappa = -\delta^\lambda_\kappa + \tilde{\xi}_\kappa \tilde{\xi}^\lambda, \quad \phi^\lambda_\mu \tilde{\xi}^\mu = 0, \quad \phi^\lambda_\mu \tilde{\xi}^\mu = 0, \quad \phi_{\mu\lambda} + \phi_{\lambda\mu} = 0$$

and

(3.14) $$D_\mu \phi^\kappa_\lambda = \tilde{\xi}_\lambda \delta^\kappa_\mu - \tilde{\xi}^\kappa g_{\mu\lambda}, \quad D_\mu \tilde{\xi}^\kappa = \phi^\kappa_\mu,$$

where $\phi_{\mu\lambda} = g_{\kappa\lambda} \phi^\kappa_\mu$. Denoting by $\mathcal{L}$ the Lie differentiation with respect to the vector field $\tilde{\xi}$, we find

(3.15) $$\mathcal{L} \phi^\lambda_\mu = 0.$$

Putting in each U

(3.16) $$\phi_j^i = \phi_\mu^\lambda E_j^\mu E_\lambda^i,$$

we can see that ϕ_j^i defines a global tensor field of the same type as that of ϕ_j^i, which will be denoted by the same letter, with the help of (3.15), $\mathcal{L}E_j^\mu = 0$ and $\mathcal{L}E_\lambda^i = 0$. Moreover, using (3.9), (3.14) and (3.16), we easily see

(3.17) $$\phi_j^i = -h_j^i,$$

which satisfies

(3.18) $$\phi_j^i \phi_h^j = -\delta_h^i.$$

Differentiating (3.16) covariantly along CP^n and using (3.9) and (3.14), we have

(3.19) $$\tilde{\nabla}_j \phi_h^i = 0,$$

where $\tilde{\nabla}$ denotes the projection of D. Hence the base space CP^n admits a Kaehlerian structure $\{\phi_j^i,\ g_{ji}\}$ which is represented by the structure tensor h_j^i of the submersion $\tilde{\pi} : S^{2n+1} \longrightarrow CP^n$ defined by the Hopf-fibration.

Let's denote by $K_{\kappa\mu\nu}{}^\lambda$ and $K_{kji}{}^h$ components of the curvature tensors of $(S^{2n+1}, g_{\lambda\mu})$ and (CP^n, g_{ji}) respectively. Since the unit sphere S^{2n+1} is a space of constant curvature 1, using the equations of co-Gauss, we have

$$K_{kji}{}^h = K_{\kappa\mu\nu}{}^\lambda E_k^\kappa E_j^\mu E_i^\nu E_\lambda^h + h_k^h h_{ji} - h_j^h h_{ki} - 2h_{kj} h_i^h$$

and together with (3.17)

$$K_{kji}{}^h = \delta_k^h g_{ji} - \delta_j^h g_{ki} + \phi_k^h \phi_{ji} - \phi_j^h \phi_{ki} - 2\phi_{kj} \phi_i^h.$$

Hence CP^n is a Kaehlerian manifold with constant holomorphic sectional curvature 4 (Cf. Ishihara and Konishi [2]). Putting

(3.20) $$\begin{cases} \phi_i^j B_b^i = \phi_a^b B_a^j + \phi_b^x N_x^j, \\ \phi_i^j N_x^i = -\phi_x^a B_a^j + \phi_x^y N_y^j, \end{cases}$$

as already shown in section 2, we can easily find the algebraic relations (2.13) $\sim$(2.17) and the structure equations (2.18)$\sim$(2.24) with $c=4$ which will be very useful.

Now we put in each nerghborhood $\overline{U}$ of $\overline{M}$

(3.21) $$\phi_\beta^\alpha = \phi_b{}^a E_\beta^b E_a^\alpha,\quad \phi_x^\alpha = \phi_x^a E_a^\alpha,\quad \phi_\alpha^x = \phi_a^x E_\alpha^a,$$

where, here and in the sequel, we denote the lifts of functions by the same letters as those the given functions. Then, using (3.4), (3.8), (3.20) and (3.21) and taking account of $N_x^\kappa = N_x^j E_j^\kappa$, we obtain

$$\phi_\mu^\kappa B_\alpha^\mu = \phi_\alpha^\beta B_\beta^\kappa + \phi_\alpha^x N_x^\kappa, \tag{3.22}$$

$$\phi_\mu^\kappa N_x^\mu = -\phi_x^\alpha B_\alpha^\kappa + \phi_x^y N_y^\kappa. \tag{3.23}$$

Transvecting ϕ_κ^λ to (3.22) and (3.23) respectively and using (3.13), (3.22) and (3.23) in the usual way, we can easily obtain that

$$\begin{gathered} \phi_\alpha^\gamma \phi_\gamma^\beta - \phi_\alpha^x \phi_x^\beta - \xi_\alpha \xi^\beta = -\delta_\alpha^\beta, \\ \phi_\alpha^\beta \phi_\beta^x + \phi_\alpha^y \phi_y^x = 0, \quad \phi_x^\beta \phi_\beta^\alpha + \phi_x^y \phi_y^\alpha = 0, \\ \phi_x^z \phi_z^y - \phi_x^\alpha \phi_\alpha^y = -\delta_x^y, \\ \phi_\alpha^\beta \xi_\beta = 0, \quad \xi^\alpha \phi_\alpha^\beta = 0, \quad \phi_\alpha^x \xi^\alpha = 0, \quad \xi_\alpha \phi_x^\alpha = 0, \\ \phi_{\beta\alpha} = -\phi_{\alpha\beta}, \quad \phi_{\alpha x} = \phi_{x\alpha}, \quad \phi_{xy} = -\phi_{yx}, \end{gathered} \tag{3.24}$$

where we have put $\phi_{\beta\alpha} = \phi_\beta^\gamma g_{\alpha\gamma}$, $\phi_{\alpha x} = \phi_\alpha^y g_{yx}$, $\phi_{x\alpha} = \phi_x^\beta g_{\beta\alpha}$ and $\phi_{xy} = \phi_x^z g_{zy}$.

Applying the operator $\tilde{\nabla}_\gamma = B_\gamma^\kappa D_\kappa$ to (3.22) and (3.23) respectively and making use of (3.11), (3.14) (3.22) and (3.23), we also find

$$\begin{gathered} \tilde{\nabla}_\gamma \phi_\beta^\alpha = \xi_\beta \delta_\gamma^\alpha - \xi^\alpha g_{\gamma\beta} + A_{\gamma x}^\alpha \phi_\beta^x - A_{\gamma\beta}{}^x \phi_x^\alpha, \\ \tilde{\nabla}_\beta \phi_\alpha^x = A_{\beta\alpha}{}^y \phi_y^x - A_{\beta\gamma}{}^x \phi_\alpha^\gamma, \quad \tilde{\nabla}_\beta \phi_x^\alpha = A_{\beta x}^\gamma \phi_\gamma^\alpha - A_{\beta y}^\alpha \phi_x^y, \\ \tilde{\nabla}_\beta \phi_x^y = A_{\beta\alpha}{}^y \phi_x^\alpha - A_{\beta x}^\alpha \phi_\alpha^y. \end{gathered} \tag{3.25}$$

Also, applying the operator $\tilde{\nabla}_\beta$ to (3.5) and taking account of (3.11) and (3.14), we have

$$\tilde{\nabla}_\beta \xi^\alpha = \phi_\beta^\alpha, \quad A_{\beta\alpha}{}^x \xi^\alpha = \phi_\beta^x, \quad A_{\beta x}^\alpha \xi^\beta = \phi_x^\alpha, \tag{3.26}$$

which and (3.9) and (3.21) imply

$$\phi_b^a = -h_b^a. \tag{3.27}$$

Moreover, in such a submanifold M, its Ricci equation is given by

$$K_{\beta\alpha y}{}^x = A_{\beta\gamma}{}^x A_{\alpha y}{}^\gamma - A_{\alpha\gamma}{}^x A_{\beta y}^\gamma \tag{3.28}$$

because the ambient manifold S^{2n+1} is a space of constant curvature.

Now we apply the operator $\nabla_b = B_b^j \tilde{\nabla}_j = E_b^\alpha \tilde{\nabla}_\alpha$ to (3.4). Then, using (3.11) and

(3.12), we have

$$A_{ba}{}^{x}N^{j}_{x}E^{a}_{\alpha}+B^{j}_{a}E^{\beta}_{b}\tilde{\nabla}_{\beta}E^{a}_{\alpha}=B^{i}_{b}E^{\mu}_{i}(D_{\mu}E^{j}_{\kappa})B^{\kappa}_{\alpha}+E^{j}_{\kappa}E^{\beta}_{b}A_{\beta\alpha}{}^{x}N^{\kappa}_{x},$$

from which taking account of (3.9), (3.10) and (3.27),

$$A_{ba}{}^{x}N^{j}_{x}E^{a}_{\alpha}-\phi^{a}_{b}B^{j}_{a}\xi_{\alpha}=-\phi^{j}_{i}B^{i}_{b}\xi_{\alpha}+(A_{\beta\alpha}{}^{x}E^{\beta}_{b})N^{j}_{x},$$

or using (3.20),

$$A_{\beta\alpha}{}^{x}E^{\beta}_{b}=A_{ba}{}^{x}E^{a}_{\alpha}+\phi^{x}_{b}\xi_{\alpha}. \tag{3.29}$$

Transvecting (3.29) with E^{b}_{γ} and changing the index γ with β, we get

$$A_{\beta\alpha}{}^{x}=A_{ba}{}^{x}E^{b}_{\beta}E^{a}_{\alpha}+\xi_{\beta}\phi^{x}_{\alpha}+\xi_{\alpha}\phi^{x}_{\beta} \tag{3.30}$$

with the help of (3.21) and (3.26).

Applying the operator $\nabla_{c}=E^{\gamma}_{c}\tilde{\nabla}_{\gamma}$ to (3.30), we have

$$E^{\gamma}_{c}\tilde{\nabla}_{\gamma}A_{\beta\alpha}{}^{x}=(\nabla_{c}A_{ba}{}^{x})E^{b}_{\beta}E^{a}_{\alpha}+A_{ba}{}^{x}E^{\gamma}_{c}(\tilde{\nabla}_{\gamma}E^{b}_{\beta})E^{a}_{\alpha}+A_{ba}{}^{x}E^{b}_{\beta}E^{\gamma}_{c}\tilde{\nabla}_{\gamma}E^{a}_{\alpha}$$
$$+E^{\gamma}_{c}(\tilde{\nabla}_{\gamma}\xi_{\beta})\phi^{x}_{\alpha}+\xi_{\beta}E^{\gamma}_{c}\tilde{\nabla}_{\gamma}\phi^{x}_{\alpha}+E^{\gamma}_{c}(\tilde{\nabla}_{\gamma}\phi^{x}_{\beta})\xi_{\alpha}+\phi^{x}_{\beta}E^{\gamma}_{c}\tilde{\nabla}_{\gamma}\xi_{\alpha},$$

from which, substituting (3.10) with $h^{a}_{b}=-\phi^{a}_{b}$, (3.25) and (3.26),

$$E^{\gamma}_{c}\tilde{\nabla}_{\gamma}A_{\beta\alpha}{}^{x}=(\nabla_{c}A_{ba}{}^{x})E^{b}_{\beta}E^{a}_{\alpha}-A_{ba}{}^{x}\phi^{b}_{c}(\xi_{\beta}E^{a}_{\alpha}+\xi_{\alpha}E^{a}_{\beta})+\phi_{\gamma\beta}E^{\gamma}_{c}\phi^{x}_{\alpha}+\phi_{\gamma\alpha}E^{\gamma}_{c}\phi^{x}_{\beta}$$
$$+\xi_{\beta}E^{\gamma}_{c}(A_{\gamma\alpha}{}^{y}\phi^{x}_{y}-A_{\gamma\delta}{}^{x}\phi^{\delta}_{\alpha})+\xi_{\alpha}E^{\gamma}_{c}(A_{\gamma\beta}{}^{y}\phi^{x}_{y}-A_{\gamma\delta}{}^{x}\phi^{\delta}_{\beta}),$$

or using (3.21) and (3.29),

$$E^{\gamma}_{c}\tilde{\nabla}_{\gamma}A_{\beta\alpha}{}^{x}=(\nabla_{c}A_{ba}{}^{x}+\phi_{cb}\phi^{x}_{a}+\phi_{ca}\phi^{x}_{b})E^{b}_{\beta}E^{a}_{\alpha}-(A_{ba}{}^{x}\phi^{b}_{c}+A_{bc}{}^{x}\phi^{b}_{a}$$
$$-A_{ca}{}^{y}\phi^{x}_{y})(\xi_{\beta}E^{a}_{\alpha}+E^{a}_{\beta}\xi_{\alpha})+2(\phi^{y}_{c}\phi^{x}_{y})\xi_{\beta}\xi_{\alpha}. \tag{3.31}$$

4. Anti-invariant submanifold of CP^{n}

If the transformation ϕ^{i}_{j} of any vector tangent to M is orthogonal to M, the submanifold M is said to be *anti-invariant* to CP^{n}. Then at any point $P\in M$ we have

$$\phi(T_{p}(M))\perp T_{p}(M),$$

and consequently

$$\phi^{a}_{b}=0 \tag{4.1}$$

in the sense of (3.20).

In this section we shall consider such a submanifold M of CP^{n} that at any point $P\in M$ we have $\phi(T_p(M))\perp T_p(M)$. Then we first find from (2.16) and (3.21)

(4.2) $$\phi_x^y\phi_y^b=0,$$

(4.3) $$\phi_\beta^\alpha=0$$

respectively. By means of (3.22) and (4.3) we can see that the submanifold $\overline{M}$ is also anti-invariant in S^{2n+1} in the sense of (3.22).

Now we assume that the second fundamental tensor of M is parallel, i.e., $\nabla_c A_{ba}{}^x=0$ and that the normal bundle $N(M)$ of M is trivial. Then (2.24) with $c=4$ and $\phi_b^a=0$ imply

$$\phi_b^x\phi_{ay}-\phi_a^x\phi_{by}+A_{be}{}^xA_{ay}^e-A_{ae}{}^xA_{by}^e=0,$$

from which, differentiating covariantly and using (2.19) and $\nabla_c A_{ba}{}^x=0$, we find

$$A_{cb}{}^z\phi_z^x\phi_{ay}+\phi_b^xA_{ca}{}^z\phi_{zy}-A_{ca}{}^z\phi_z^x\phi_{by}-\phi_a^xA_{cb}{}^z\phi_{zy}=0.$$

Transvecting the above equation with ϕ_x^a and using (4.2), we obtain $2(n-1)A_{cb}{}^x\phi_x^y=0$, which implies

(4.4) $$A_{cb}{}^x\phi_x^y=0$$

and consequently

(4.5) $$\nabla_b\phi_a^x=0,\ \nabla_b\phi_x^a=0$$

with the help of (2.19).

We differentiate (4.2) covariantly along M. Then we have by using (4.5)

$$(\nabla_d\phi_x^y)\phi_y^c=0,$$

from which, transvecting with ϕ_c^z and taking account of (2.17),

$$\nabla_d\phi_x^z+(\nabla_c\phi_x^y)\phi_y^w\phi_w^z=0.$$

On the other hand $(\nabla_c\phi_x^y)\phi_y^w\phi_w^z=0$ because of (2.20), (4.2) and (4.4). Hence we have

(4.6) $$\nabla_d\phi_x^y=0.$$

THEOREM 1. *Let M be an anti-invariant submanifold of a complex projective space CP^n and $\pi:\overline{M}\longrightarrow M$ the submersion which is compatible with the Hopf-fibration $\tilde{\pi}:S^{2n+1}\longrightarrow CP^n$. If the second fundamental form of M is parallel and* *the normal connection is flat, then the second fundamental form of $\overline{M}$ is also paralled and, moreover, the normal connection of $\overline{M}$ is flat.*

PROOF. Under the our assumption, we can easily check that

$$E_c^\gamma \tilde{\nabla}_\gamma A_{\beta\alpha}{}^x = 0$$

because of (3.31), (4.1), (4.2) and (4.4). Transvecting the above equation with E_δ^c gives

$$\tilde{\nabla}_\delta A_{\beta\alpha}{}^x = \xi_\delta \xi^\gamma \tilde{\nabla}_\beta A_{\gamma\alpha}{}^x \tag{4.7}$$

because $\tilde{\nabla}_\gamma A_{\beta\alpha}{}^x - \tilde{\nabla}_\beta A_{\gamma\alpha}{}^x = 0$.

On the other hand, differentiating the second equation of (3.26) covariantly and using (3.26) and (4.3), we obtain

$$(\tilde{\nabla}_\beta A_{\alpha\gamma}{}^x)\xi^\gamma = \tilde{\nabla}_\beta \phi_\alpha^x,$$

from which, taking account of (3.10) with $h_c^a = \phi_c^a = 0$, (3.21) and (4.5), we can easily find

$$(\tilde{\nabla}_\beta A_{\alpha\gamma}{}^x)\xi^\gamma = 0. \tag{4.8}$$

Hence, from (4.7) and (4.8), we have

$$\tilde{\nabla}_\gamma A_{\beta\alpha}^x = 0.$$

Next, in order to prove the second assertion we compute directly $K_{\gamma\beta y}{}^x$ components of the normal connection of $\overline{M}$ by using (3.28) and (3.30).

$$A_{\gamma\alpha}{}^x A_{\beta y}^\alpha = (A_{ba}{}^x E_\gamma^b E_\alpha^a + \xi_\gamma \phi_\alpha^x + \xi_\alpha \phi_\gamma^x)(A_{dy}^c E_\beta^d E_c^\alpha + \xi_\beta \phi_y^\alpha + \xi^\alpha \phi_{y\beta}),$$

which and (3.21) and (3.24) imply

$$A_{\gamma\alpha}{}^x A_{\beta y}^\alpha = A_{be}{}^x A_{dy}^e E_\gamma^b E_\beta^d + A_{ba}{}^x \phi_y^a E_\gamma^b \xi_\beta + A_{dy}^c \phi_c^x \xi_\gamma E_\beta^d + (\phi_\alpha^x \phi_y^\alpha)\xi_\gamma \xi_\beta + (\phi_b^x \phi_{yd}) E_\gamma^b E_\beta^d,$$

and consequently

$$A_{\gamma\alpha}^x A_{\beta y}^\alpha - A_{\beta\alpha}^x A_{\gamma y}^\alpha = (A_{be}^x A_{dy}^e - A_{de}^x A_{by}^e + \phi_b^x \phi_{yd} - \phi_d^x \phi_{yb}) E_\gamma^b E_\beta^d$$
$$+ (A_{ba}^x \phi_y^a - A_{by}^a \phi_a^x)(E_\gamma^b \xi_\beta - \xi_\gamma E_\beta^b).$$

Hence we have

$$K_{\gamma\beta y}{}^x = K_{bdy}{}^x E_\gamma^b E_\beta^d + (\nabla_b \phi_y^x)(E_\gamma^b \xi_\beta - \xi_\gamma E_\beta^b)$$

if the submanifold is anti-invariant in CP^n, which and (4.6) imply our last assertion. Thus we complete the proof of the theorem.

Combining Theorem A and Theorem B, we have

THEOREM 2. *Let M be a compact orientable anti-invariant submanifold of a complex projective space CP^n of a real codimension p and $\pi : \overline{M} \longrightarrow M$ the submersion which is compatible with the Hopf-fibration $\tilde{\pi} : S^{2n+1} \longrightarrow CP^n$. If the second fundamental form of M is parallel and the normal connection is flat. then*

$$M = S^1(r_1) \times \cdots \times S^1(r_{2n+1-p})/\sim,$$

where $r_1^2 + \cdots + r_{2n+1-p}^2 = 1$.

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