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ON DOUBLE INFINITE SERIES INVOLVING THE H-FUNCTION OF TWO VARIABLES

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0. Abstract

In this paper, we obtain two new double infinite series for the H-function of two variables, by which we also obtain a single infinite series involving the H-function of two variables. On account of the most general nature of the H-functin of two variables, a number of related double infinite series for simpler functions follow as special cases of our results. As an illustration, we obtain here from one of our main series, the corresponding series for Kampé de Fériet function and Fox's H-function. A number of other series involving a very large spectrum of special functions also follow as special cases of our main series but we are not recording them here for want of space.

1. Introduction

The H-function of two variables occurring in this paper is defined and represented in terms of double Mellin-Barnes type contour integral in the following manner [6, p. 117] :

$$H[x,y]=H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, & n_1 \\ p_1, & q_1 \end{pmatrix} & \begin{array}{l}(a_j;\alpha_j,A_j)_{1,p_1}\\ (b_j;\beta_j,B_j)_{1,q_1}\end{array} & \\ \begin{pmatrix} m_2, & n_2 \\ p_2, & q_2 \end{pmatrix} & \begin{array}{l}(c_j,\gamma_j)_{1,p_2}\\ (d_j,\delta_j)_{1,q_2}\end{array} & \begin{array}{c} x \\ y\end{array} \\ \begin{pmatrix} m_3, & n_3 \\ p_3, & q_3 \end{pmatrix} & \begin{array}{l}(e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3}\end{array} & \end{array}\right]$$

$$=\left(\frac{1}{2\pi i}\right)^2\int_{L_1}\int_{L_2}\varphi(s,t)\theta_1(s)\theta_2(t)x^s y^t\,ds\,dt \tag{1.1}$$

where

$$\varphi(s,t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)}{\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j s+B_j t)\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)} \tag{1.2}$$

$$\theta_1(s)=\frac{\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j s)\prod_{j=1}^{n_2}\Gamma(1-c_j+r_j s)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j s)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-r_j s)} \tag{1.3}$$

and with $\theta_2(t)$ defined analogously in terms of the parameters (e_j, E_j) and (f_j, F_j).

The nature of contours L_1 and L_2 involved in (1.1) and the conditions under which $H[x, y]$ defined in (1.1) converges and a number of simpler functions involving one and two variables which are special cases of $H[x,y]$ can be referred to in the paper by Mittal and Gupta [6].

Also,

$$H\left[\begin{array}{c|c|c} \begin{pmatrix} 0, & n_1 \\ p_1, & q_1 \end{pmatrix} & \begin{array}{c}(a_j\,;\,\alpha_j,\ A_j)_{1,\,p_1}\\ (b_j\,;\,\beta_j,\ B_j)_{1,\,q_1}\end{array} & x \\ \cdots\cdots & \cdots\cdots & \\ \cdots\cdots & \cdots\cdots & y \end{array}\right]$$

will indicate that parameters shown as $\cdots\ \cdots$ are the same as that of $H[x,y]$ in (1.1) and similarly for other such notations,

2. The following results [1, p. 25(1.1)] is required to establish our main results:

$$F^{2,1}_{0,2}\left[\begin{array}{l}\alpha,\ \alpha_1 : -m\,;\,\beta \\ \text{———} : \delta, \delta_1\,;\,1+\beta, \alpha_1-n\end{array}\right]$$

$$=\frac{\Gamma(1-\alpha)\Gamma(1+\beta)(1+\beta-\alpha_1)_n\ (\delta-\alpha+\beta)_m(\delta-\alpha_1+\beta)_m}{\Gamma(1+\beta-\alpha)(1-\alpha_1)_n(\delta)_m(1-\delta_1-m)_m}$$

where $\delta_1=1+\alpha+\alpha_1-2\beta-m-\delta$ and α_1 is either a fraction or $\mathrm{Re}(\alpha_1)>n$. (2.1)

where $F^{2,\,1}_{0,\,2}$ is the well known Kampé de Fériet function [2] for $x=y=1$ as defined below:

$$F^{p_1,\,p_2}_{q_1,\,q_2}=F^{p_1,p_2}_{q_1,q_2}\left[\begin{array}{c|c}(a_j)_{1,\,p_1} : (c_j)_{1,\,p_2}\,;\,(e_j)_{1,\,p_2} & \\ (b_j)_{1,\,q_1}\,;\,(d_j)_{1,\,q_2} : (f_j)_{1,\,q_2} & 1,1\end{array}\right]$$

$$=\sum_{m,n=0}^{\infty}\frac{\prod_{j=1}^{p_1}(a_j)_{m+n}\prod_{j=1}^{p_2}(c_j)_m\prod_{j=1}^{p_2}(e_j)_n}{\prod_{j=1}^{q_1}(b_j)_{m+n}\prod_{j=1}^{q_2}(d_j)_m\prod_{j=1}^{q_2}(f_j)_n\ m!n!} \tag{2.2}$$

3. Main results

(1)
$$\sum_{r,s=0}^{\infty}\frac{(-m)_r(\alpha)_{r+s}(\alpha^{1})_{r+s}}{r!\quad s!\quad(\alpha'-n)_s}$$
$$\times H\left[\begin{array}{c|l|c}\begin{pmatrix}0, & n_1+1\\ p_1+3, & q_1+1\end{pmatrix} & \begin{array}{l}(1-\beta-s:\rho,\sigma),(a_j:\alpha_j,A_j)_{1,p_1},(\delta'+r:\rho,\sigma),\\ (\delta+r:\rho,\sigma)\\ (b_j:\beta_j,B_j)_{1,q_1},(-\beta-s:\rho,\sigma)\end{array} & x\\ \cdots & \cdots & \\ \cdots & \cdots & y\end{array}\right]$$
$$=\frac{(-1)^m\Gamma(1-\alpha)\;(\delta-\alpha+\beta)_m\,(\delta-\alpha'+\beta)_m}{(1-\alpha')_n}$$
$$\times H\left[\begin{array}{c|l|c}\begin{pmatrix}0, n_1+2\\ p_1+4, q_1+2\end{pmatrix} & \begin{array}{l}(1-\beta:\rho,\sigma),\ (\alpha'-\beta-n:\rho,\sigma),(a_j:\alpha_j,A_j)_{1,p_1},\\ (\sigma'+m:\rho,\sigma),\ (\delta+m:\rho,\sigma)\\ (b_j:\beta_j,B_j)_{1,q_1},\ (\alpha'-\beta:\rho,\sigma),\ (\alpha-\beta:\rho,\sigma)\end{array} & \begin{array}{c}x\\ \\ y\end{array}\\ \cdots & \cdots & \\ \cdots & \cdots & \\ \cdots & \cdots & \end{array}\right]$$
where $\delta'=1+\alpha+\alpha'-2\beta-m-\delta$ (3.1)

(2)
$$\sum_{r,s=0}^{\infty}\frac{(-m)_r}{r!s!(\delta)_r}H\left[\begin{array}{c|l|c}\begin{pmatrix}0, n_1+1\\ p_1+2, q_1+1\end{pmatrix} & \begin{array}{l}(1-\beta-s:\rho,\sigma),(a_j:\alpha_j,A_j)_{1,p_1},\\ (\delta'+r:\rho,\sigma)\\ (b_j:\beta_j,B_j)_{1,q_1},(-\beta-s:\rho,\sigma)\end{array} & x\\ \begin{pmatrix}m_2, n_2+1\\ p_2+2, q_2\end{pmatrix} & \begin{array}{l}(1-\alpha-r-s,\rho),(c_j,\gamma_j)_{1,p_2},(1-\alpha,\rho)\\ (d_j,\delta_j)_{1,q_2}\end{array} & y\\ \begin{pmatrix}m_3, n_3+1\\ p_3+1, q_3+1\end{pmatrix} & \begin{array}{l}(1-\alpha'-r-s,\sigma),(e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3},(1-\alpha'+n-s,\sigma)\end{array} & \end{array}\right]$$
$$=\frac{(-1)^{m+n}}{(\delta)_m}H\left[\begin{array}{c|l|c}\begin{pmatrix}0, n_1+1\\ p_1+2, q_1\end{pmatrix} & \begin{array}{l}(1-\beta:\rho,\sigma),(a_j:\alpha_j,A_j)_{1,p_1},(\delta'+m:\rho,\sigma)\\ (b_j:\beta_j,B_j)_{1,q_1}\end{array} & \\ \begin{pmatrix}m_2, n_2+3\\ \\ p_2+3, q_2+2\end{pmatrix} & \begin{array}{l}(1-\alpha,\rho),\ (1+\alpha'-\delta-\beta-m,\ \rho),\\ (\alpha'-\beta-n,\ \rho),\ (c_j,\gamma_j)_{1,p_2},\\ (d_j,\delta_j)_{1,q_2},\ (1+\alpha'-\delta-\beta,\ \rho),(\alpha'-\beta,\ \rho)\end{array} & x\\ \begin{pmatrix}m_3, n_3+1\\ p_3+1, q_3+2\end{pmatrix} & \begin{array}{l}(1-\delta-\beta+\alpha-m,\sigma),\ (e_j,E_j)_{1,p_3}\\ (f_j,F_j)_{1,q_3},(1+\alpha-\delta-\beta,\sigma),\ (\alpha-\beta,\ \sigma)\end{array} & y\end{array}\right]$$
where $\delta'=1+\alpha+\alpha'-2\beta-m-\delta$ (3.2)

PROOF of 3.1. To prove (3.1), we express the H-function of two variables involved in the left hand side of (3.1) in terms of double Mellin-Barnes type contour integral, we get the following left hand side of (3.1) :

$$\sum_{r,\,s=0}^{\infty}\frac{(\alpha)_{r+s}(\alpha')_{r+s}(-m)_r}{r!\,s!\,(\alpha'-n)_s}\left(\frac{1}{2\pi i}\right)^2$$

$$\times\int_{L_1}\int_{L_2}\frac{\varphi(\xi,\ n)\theta_1(\xi)\theta_2(\eta)\Gamma(\beta+s+\rho\xi+\sigma\eta)}{\Gamma(1+\beta+s+\rho\xi+\sigma\eta)\Gamma(\delta'+r-\rho\xi-\sigma\eta)\Gamma(\delta+r-\rho\xi-\sigma\eta)}x^{\xi}y^{\eta}d\xi d\eta \quad \text{(A)}$$

Now on changing the order of integration and summation in (A), which is assumed to be justified, we get, the following form of (A) by using results [4, p. 3(2)] and (2.2):

$$\left(\frac{1}{2\pi i}\right)^2\int_{L_1}\int_{L_2}\frac{\varphi(\xi,\ \eta)\theta_1(\xi)\theta_2(\eta)\Gamma(\beta+\rho\xi+\sigma\eta)x^{\xi}y^{\eta}}{\Gamma(1+\beta+\rho\xi+\sigma\eta)\Gamma(\delta'-\rho\xi-\sigma\eta)\Gamma(\delta-\rho\xi-\sigma\eta)}$$

$$\times\ F^{2,2}_{0,2}\left[\begin{matrix}\alpha,\ \alpha':\ -m:\ \beta+\rho\xi+\sigma\eta \\ -\ :\delta'-\rho\xi-\sigma\eta,\ \delta-\rho\xi-\sigma\eta:\alpha'-n,\ 1+\beta+\rho\xi+\sigma\eta\end{matrix}\right]d\xi d\eta \quad \text{(B)}$$

Now, with the help of (2.1) and a known result [4, p. 3(3)], we get, the following form of (B):

$$\frac{(-1)^m\Gamma(1-\alpha)(\delta-\alpha+\beta)_m\ (\delta-\alpha'+\beta)_m}{(1-\alpha')_n}\left(\frac{1}{2\pi i}\right)^2\int_{L_1}\int_{L_2}\frac{\varphi(\xi,\ \eta)\theta_1(\xi)\theta_2(\eta)}{\Gamma(1+\beta-\alpha'+\rho\xi+\sigma\eta)}$$

$$\times\frac{\Gamma(\beta+\rho\xi+\sigma\eta)\Gamma(1+\beta-\alpha'+n+\rho\xi+\sigma\eta)x^{\xi}y^{\eta}d\xi d\eta}{\Gamma(1+\beta-\alpha+\rho\xi+\sigma\eta)\Gamma(\delta+m-\rho\xi-\sigma\eta)\Gamma(\delta'+m-\rho\xi-\sigma\eta)} \quad \text{(C)}$$

Now, on interpreting the result with the help of the definition of the H-function of two variables, we get, the right hand side of (3.1). This completes the proof.

To prove the result (3.2), we proceed exactly on the lines similar to that given in the proof of (3.1).

4. Particular cases

(1) On taking $m=0$ in (3.1), we get, the following single series involving the H-function of two variables:

$$\sum_{s=0}^{\infty}\frac{(\alpha)_s(\alpha')_s}{s!\,(\alpha'-n)_s}\,H\left[\begin{matrix}\begin{pmatrix}0,\ n_1+1\\ p_1+1,q_1+1\end{pmatrix}\\ \cdots \\ \cdots\end{matrix}\left|\begin{matrix}(1-\beta-s:\rho,\ \sigma),\ (a_j:\alpha_j,\ A_j)_{1,p_1}\\ (b_j:\beta_j,\ B_j)_{1,q_1},\ (-\beta-s:\rho,\ \sigma)\\ \cdots\\ \cdots\end{matrix}\right|\begin{matrix}x\\ y\end{matrix}\right]$$

$$=\frac{\Gamma(1-\alpha)}{(1-\alpha')_n}H\left[\begin{matrix}\begin{pmatrix}0,\ n_1+2\\ p_1+2,q_1+2\end{pmatrix}\\ \cdots \\ \cdots\end{matrix}\left|\begin{matrix}(1-\beta:\rho,\sigma),(\alpha'-\beta-n:\rho,\sigma),\ (a_j:\alpha_j,A_j)_{1,p_1}\\ (b_j:\beta_j,\ B_j)_{1,q_1},\ (\alpha'-\beta:\rho,\ \sigma),\ (\alpha-\beta:\rho,\ \sigma)\\ \cdots\\ \cdots\end{matrix}\right|\begin{matrix}x\\ y\end{matrix}\right]$$

(4.1)

(ii) Also, if we put $\rho=\sigma=1$, $n_j=p_1+2$, $n_2=n_3=p_3=p_2$, $q_3=q_2$, $m_2=m_3=1$, $d_1=f_1=0$, all α_j's, A_j's, β_j's, B_j's, r_j's, δ_j's, E_j's and F_j's equal to unity and replace q_2 by q_2+1, $(a_j)_{1,p_1}$ by $(1-a_j)_{1,p_1}$, $(b_j)_{1,q_1}$ by $(1-b_j)_{1,q_1}$, $(c_j)_{1,p_2}$ by $(1-c_j)_{1,p_2}$, $(d_j+1)_{1,q_2}$ by $(1-d_j+1)_{1,q_2}$, $(e_j)_{1,p_2}$ by $(1-e_j)_{1,p_2}$, $(f_j+1)_{1,q_2}$ by $(1-f_j+1)_{1,q_2}$, x by $-x$ and y by $-y$ in (3.1), we get, after a little simplification, the following double infinite series involving Kampé de Fériet function [2] :

$$\sum_{r,\, s=0}^{\infty} \frac{(-m)_r(\alpha)_{r+s}(\alpha')_{r+s}(\beta)_s}{s!\; s!(\alpha'-n)_s(\delta')_r(\delta)_s(1+\beta)_s} \cdot F^{p_1+3,\; p_2}_{q_1+1,\; q_2}\left[\begin{matrix}\beta+s, 1-\delta-r,\; 1-\delta'-r, \\ 1+\beta+s, (b_j)_{1,q_1} \quad :\end{matrix}\right.$$
$$\left.\begin{matrix}(a_j)_{1,p_1} : (c_j)_{1,p_2} : (e_j)_{1,p_2} \\ (d_j)_{1,q_2} : (f_j)_{1,q_2}\end{matrix}\;\middle|\; x,\ y\right]$$
$$=\frac{\Gamma(1+\beta)(-1)^m\Gamma(1-\alpha)(\delta-\alpha+\beta)_m(\delta-\alpha'+\beta)_m(1-\alpha'+\beta)_n}{(1-\alpha')_n(\delta)_m(\delta')_m\Gamma(1-\alpha+\beta)}$$
$$\times F^{p_1+4,\; p_2}_{q_1+2,\; q_2}\left[\begin{matrix}\beta,\; 1-\alpha'+\beta+n,\; 1-\delta-m,\; 1-\delta'-m,\; (a_j)_{1,p_1} : (c_j)_{1,p_2} : (e_j)_{1,p_2} \\ 1-\alpha'+\beta,\; 1-\alpha+\beta,\; (b_j)_{1,q_1} : (d_j)_{1,q_2} : (f_j)_{1,q_2}\end{matrix}\;\middle|\; x,\ y\right] \tag{4.2}$$

(iii) Again, if we put $m_3=1$, $f_1=0$, $\sigma=1$, all A_j's, B_j's, E_j's and F_j's equal to unity, replace q_3 by q_3+1 and let $y\to 0$ in (3.1), by virtue of [5, p. 123(3.5)], we get, after a little simplification, the following series involving Fox's H-function:

$$\sum_{r,\, s=0}^{\infty} \frac{(-m)_r(\alpha)_{r+s}(\alpha')_{r+s}}{s!s!(\alpha'-n)_s} H^{m_2,\; n_2+1}_{p_2+3,\; q_2+1}\left[x \;\middle|\; \begin{matrix}(1-\beta-s,\ \rho),\ (a_j,\ \alpha_j)_{1,p_2}, \\ (\delta'+r,\ \rho),\ (\delta+r,\ \rho) \\ (b_j,\ \beta_j)_{1,q_2},\ (-\beta-s,\ \rho).\end{matrix}\right]$$
$$=\frac{(-1)^m\Gamma(1-\alpha)(\delta-\alpha+\beta)_m(\delta-\alpha'+\beta)_m}{(1-\alpha')_m}$$
$$\times H^{m_2, n_2+2}_{p_2+4, q_2+2}\left[x\;\middle|\; \begin{matrix}(1-\beta,\ \rho),\ (\alpha'-\beta-n,\ \rho),\ (a_j,\ \alpha_j)_{1,p_2},\ (\delta+m,\ \rho),\ (\delta'+m,\ \rho) \\ (b_j,\ \beta_j)_{1,q_2},\ (\alpha'-\beta,\ \rho),\ (\alpha-\beta,\ \rho)\end{matrix}\right] \tag{4.3}$$

Similarly, we can obtain double infinite series from (3.2), as in the case of (3.1) by reducing the H-function of two variables. Also, related infinite series for other special functions which are not mentioned in the particular cases of (3.1) can be obtained from (3.1) and (3.2) by reducing the H-function of two variables into some other simpler functions, but we do not record them here explicitely for lack of space.

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