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# SOME RESULTS INVOLVING THE MULTIPLE $H$ FUNCTION

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## 0. Abstract

The object of the present paper is to obtain certain results involving the $H$ function of several complex variables. An integral involving the generalized Whittaker functions and the multiple $H$ function has been evaluated and this result has been further utilized in finding out an expansion formula for the multiple $H$ function in terms of the Laguerre polynomials. Some particular cases of interest have also been indicated.

## 1. Introduction

Throughout this paper we employ the abbreviation $(a)$ to denote the sequence of $A$ parameters $a_1, \cdots, a_A$; for each $i=1,\cdots,n$, $(b^{(i)})$ abbreviates the sequence of $B^{(i)}$ parameters $b_j^{(i)}$, $j=1,\cdots, B^{(i)}$, with similar interpretations for $(c)$, $(d^{(i)})$, etc., $i=1,\cdots,n$, it will be understood, for example, that $b^{(1)}=b'$, $b^{(2)}=b''$, and so on. Also, for the sake of brevity, we use the contracted notation

$$(1.1)\qquad [(a)]_n=\prod_{j=1}^{A}[a_j]_n,\quad [b^{(i)}]_n=\prod_{j=1}^{B(i)}[b_j^{(i)}]_n,\ i=1,\cdots,n;\ \text{etc.}$$

The multiple $H$ function defined recently by Srivastava and Panda [6] shall be defined and represented by means of the multiple contour integral as follows:

$$(1.2)\quad H[z_1,\cdots,z_n]=H^{0,\lambda:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A,C:[B',D'];\cdots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\cdots,\theta^{(n)}]:[(b'),\phi'];\cdots;|(b^{(n)}),\phi^{(n)}|;\\ [(c):\Psi',\cdots,\Psi^{(n)}]:[(d'),\delta'];\cdots;[(d^{(n)},\delta^{(n)}];\end{matrix}\ z_1,\cdots,z_n\right)$$

$$=\left(\frac{1}{2\pi W}\right)^n\int_{-w^\infty}^{w^\infty}\cdots\int_{-w^\infty}^{w^\infty}\phi_1(\rho_1)\cdots\phi_n(\rho_n)\ \Psi(\rho_1,\cdots\rho_n)\ z_1^{\rho_1}\cdots z_n^{\rho_n}\ d\rho_1\cdots d\rho_n,$$

where $\quad w=\sqrt{-1}$,

$$(1.3)\quad \phi_i(\rho_i)=\frac{\prod_{j=1}^{\mu^{(i)}}\Gamma(d_j^{(i)}-\delta_j^{(i)}\rho_i)\prod_{j=1}^{v^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\rho_i]}{\prod_{j=\mu^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\rho_i)\prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma[b_j^{(i)}-\phi_j^{(i)}\rho_i]},$$

$$\forall i\in\{1,\cdots,n\};$$

$$(1.4)\quad \Psi(\rho_1,\cdots,\rho_n)=\frac{\prod_{j=1}^{\lambda}\Gamma\left[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\rho_i\right]}{\prod_{j=\lambda+1}^{A}\Gamma\left[a_j-\sum_{i=1}^{n}\theta_j^{(i)}\rho_i\right]\prod_{j=1}^{C}\Gamma\left[1-c_j+\sum_{i=1}^{n}\Psi_j^{(i)}\rho_i\right]},$$

an empty product is interpreted as 1, the coefficients $\theta_j^{(i)}$, $j=1,\cdots,A$; $\phi_j^{(i)}$, $j=1,\cdots,B^{(i)}$; $\Psi_j^{(i)}$, $j=1,\cdots,C$; $\delta_j^{(i)}$, $j=1,\cdots,D^{(i)}$, $\forall i\in\{1,\cdots,n\}$, are positive numbers, and $\lambda$, $\mu^{(i)}$, $v^{(i)}$, $A$, $B^{(i)}$, $C$, $D^{(i)}$ are integers such that $0\le\lambda\le A$, $0\le\mu^{(i)}\le D^{(i)}$, $C\ge0$, $0\le v^{(i)}\le B^{(i)}$, $\forall i\in\{1,\cdots,n\}$. The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma[d_j^{(i)}-\delta_j^{(i)}\rho_i]$, $j=1,\cdots,\mu^{(i)}$ are seperated from the poles of $\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\rho_i]$, $j=1,\cdots,v^{(i)}$, and $\Gamma\left[1-a_j+\sum_{i=1}^{n}\theta_j^{(i)}\rho_i\right]$, $i=1,\cdots,\lambda$. When $z_1,\cdots,z_n$ are not equal to zero, the multiple integral in (1.2) converges absolutely if

$$(1.5)\quad |\arg(z_i)|<\tfrac{1}{2}\pi\,\Delta_i,\quad \forall i\in\{1,\cdots,n\},$$

where

$$(1.6)\quad \Delta_i=\sum_{j=1}^{\lambda}\theta_j^{(i)}-\sum_{j=\lambda+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{v^{(i)}}\phi_j^{(i)}-\sum_{j=v^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{\mu(i)}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,\quad \forall i\in\{1,\cdots,n\}.$$

The conditions corresponding to the aforementioned ones will be assumed to hold throughout this paper. For convinience, the case $\lambda=0$ of (1.2) shall be abbreviated as $H_1[z_1,\cdots,z_n]$. In what follows the product $\Gamma(a+b)\Gamma(a-b)$ shall be denoted by $\Gamma(a\pm b)$. The Gauss multiplication formula for the Gamma function is given by [2, p. 4]

$$(1.7)\qquad \Gamma(mz)=(2\pi)^{(m-1)/2}\,m^{(2mz-1)/2}\prod_{i=1}^{m-1}\Gamma\left(z+\frac{i}{m}\right);$$

$m$ being a positive integer.

## 2. Evaluation of the integral

The main result to be established here is

$$(2.1)\qquad \int_0^\infty x^{\beta-1}\, W_{k,e}(x)\, W_{g,p}(x)\, H_1[u_1x^{h_1},\cdots,u_n\,x^{h_n}]\,dx$$

$$=\frac{\Gamma(-2p)}{\Gamma\left(\frac{1}{2}-g-p\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}-g+p\right)^r}{r!\,(2p+1)_r}\,H^{0,2:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A+2,C+1:[B',D'];\cdots;[B^{(n)},D^{(n)}]}\left(S\right)$$

$$+\frac{\Gamma(2p)}{\Gamma\left(\frac{1}{2}-g+p\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}-g-p\right)_r}{r!\,(1-2p)_r}\,H^{0,2:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A+2,C+1:[B',D'];\cdots;[B^{(n)},D^{(n)}]}\left(T\right);$$

***valid for*** $\mathrm{Re}\left[\beta+\sum_{t=1}^{n} h_t\,\alpha_t\right]+1>|\mathrm{Re}(e)|+|\mathrm{Re}(g)|,\ \beta>0,\ |\arg(u_i)|$

$$<\frac{1}{2}\pi\Delta_i,\ \Delta_i>0;\ h_t>0,\ \alpha_t=d_j^{(t)}/\delta_j^{(t)},\ \forall t\in\{1,\cdots,n\},$$

$$S=\left(\begin{matrix}[(\pm e-\beta-p-r):h_1,\cdots,h_n],\ [(a):\theta',\cdots,\theta^{(n)}]: \\ \left[\left(k-\frac{1}{2}-p-\beta-r\right):h_1,\cdots,h_n\right],\ [(c):\Psi',\cdots,\Psi^{(n)}]:\end{matrix}\begin{matrix}[(b'),\phi'];\cdots;[(b^{(n)}),\ \phi^{(n)}]; \\ [(d'),\delta'];\cdots;[(d^{(n)}),\ \delta^{(n)}];\end{matrix}\ u_1,\cdots,u_n\right);$$

$$T=\left(\begin{matrix}[(\pm e-\beta-r+p);h_1,\cdots,h_n],\ [(a):\theta',\cdots,\theta^{(n)}]: \\ \left[\left(k-\beta-r+p-\frac{1}{2}\right);h_1,\cdots,h_n\right],\ [(c);\Psi',\cdots,\Psi^{(n)}]:\end{matrix}\begin{matrix}[(b'),\ \phi'];\cdots;[(b^{(n)}),\phi^{(n)}]; \\ [(d'),\delta'];\cdots;[(d^{(n)}),\ \delta^{(n)}]:\end{matrix}\ u_1,\cdots,u_n\right).$$

PROOF. In the integrand of (2.1), we replace the multiple $H$ function by its Mellin-Barnes contour integral (1.2) and then change the order of integration, which is justified due to the absolute convergence of the integrals involved in the process. Next we evaluated the inner integral by making use of a known relation [3, p. 410] and further interpret the results with (1.2) and (1.7) to obtain the desired result.

In particular on taking $2g=2p+1$, $2e=\alpha$, $2k=2m+\alpha+1$ in (2.1) and on using the known relations [3, p. 432], we derive the result

$$(2.2)\qquad \int_0^\infty e^{-x}\, x^{\beta+m+\frac{1}{2}\alpha}\, L_m^{(\alpha)}(x)\, H_1[u_1x^{h_1},\cdots,u_nx^{h_n}]\,dx$$

$$=\frac{(-1)^m}{m!}\,H^{0,2:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A+2,C+1:[B',D'];\cdots;[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix}\left[\left(\pm\frac{1}{2}\alpha-\beta-p\right):h_1,\cdots,h_n\right],\ [(a):\theta',\cdots,\theta^{(n)}]:\\ \left[\left(\frac{1}{2}\alpha+m-\beta-p\right):h_1,\cdots,h_n\right],\ [(c):\Psi',\cdots,\Psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'),\phi'];\cdots;[(b^{(n)}),\ \phi^{(n)}]:\\ [(d'),\delta'];\cdots;[(d^{(n)}),\delta^{(n)}]:\end{matrix}\ u_1,\cdots,u_n\right),$$

valid for

$$\mathrm{Re}\left[\beta+p+\sum_{t=1}^{n}h_t\sigma_t\right]>\frac{1}{2}|\mathrm{Re}(\alpha)|-1,\ \sigma_t=d_j^{(t)}/\delta_j^{(t)},\ h_t>0:$$

$$\forall t\in\{1,\cdots,n\},\ \Delta_i>0,\ |\arg(u_i)|<\frac{1}{2}\pi\Delta_i.$$

**3. Expansion formula**

*The expansion formula to be established here is*

(3.1) $$x^w H_1[u_1x^{h_1},\cdots,u_nx^{h_n}]=\sum_{r=0}^{\infty}\frac{(-1)^r L_r^{(\alpha)}(x)}{\Gamma(1+\alpha+r)}$$

$$H^{0,2:(\mu',\nu');\cdots;(\mu^{(n)},\nu^{(n)})}_{A+2,C+1:[B',D'];\cdots;[B^{(n)},D^{(n)}]}$$

$$\left(\begin{matrix}[(-w):h_1,\cdots,h_n],[(-\alpha-w):h_1,\cdots,h_n],[(a):\theta',\cdots,\theta^{(n)}]:\\ [(c):\Psi',\cdots,\Psi^{(n)}],[(r-w):h_1,\cdots,h_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'),\phi']:\cdots;[(b^{(n)}),\phi^{(n)}]:\\ [(d'),\delta'];\cdots;[(d^{(n)}),\delta^{(n)}]:\end{matrix}\ u_1,\cdots,u_n\right),$$

*valid for* $\mathrm{Re}\,\alpha>-1,\ \Delta_i>0,\ |\arg(u_i)|<\frac{1}{2}\pi\Delta_i,$

$$\mathrm{Re}\left[w+\sum_{t=1}^{n}h_t\sigma_t\right]>-1:h_t>0,\ \sigma_t=d_j^{(t)}/\delta_j^{(t)},\ \forall t\in\{1,\cdots,n\},$$

$$j=1,\cdots,\mu^{(i)}.$$

PROOF. Let

(3.2) $$x^w H_1[u_1x^{h_1},\cdots,u_nx^{h_n}]=\sum_{r=0}^{\infty}D_rL_r^{(\alpha)}(x)=f(x),\ \text{say}.$$

Here (3.2) is valid since $f(x)$ is continuous and of bounded variation in the interval $(0,\infty)$. On multiplying both sides of (3.2) by $x^\alpha e^{-x}L_m^{(\alpha)}(x)$ and integrating with respect to $x$ between the limits 0 and $\infty$, we have

$$(3.3)\quad \int_0^\infty x^{\alpha+w}\, e^{-x} L_m^{(\alpha)}(x)\; H_1[u_1x^{h_1},\cdots,u_nx^{h_n}]\; dx$$

$$=\sum_{r=0}^{\infty} D_r \int_0^\infty x^\alpha e^{-x} L_m^{(\alpha)}(x) L_r^{(\alpha)}(x)\, dx.$$

Now on using the orthogonality property of the Laguerre polynomials [3, p. 292] and the result (2.2), we easily get the value of $D_n$; and finally on substituting this value of $D_n$ in (3.2) we immediately arrive at the desired result.

## 4. Particular cases

On specializing the parameters of the multiple $H$-function and the generalized Whittaker functions in the results (2.1), (2.2) and (3.1), we deduce various known formulae given earlier by Bajpai [1], Saxena [4], Srivastava and Joshi [5] and various others.

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