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A GENERAL DIFFERENTIAL EQUATION FOR CLASSICAL POLYNOMIALS

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1. Introduction

The object of the present paper is to derive the two differential equations satisfied by the polynomial set $\overline{R}_n(x,y)$ and the general solutions thereof. $\overline{R}_n(x,y)$, the generalization of as many as twenty two classical polynomials such as Laguerre polynomials, Hermite polynomials, Legendre polynomials, Jacobi polynomials, Bedient polynomials etc. has been defined by us by means of the generating relation

$$(1.1) \qquad \sum_{n=0}^{\infty} \overline{R}_n(x,y)t^n = \frac{QJ\nu_1(\alpha_1 x^{m_1} y^{m_2} t^{m_3})}{(1-vx^{-m} t^{m_4})^{\alpha}}$$

$$\times H^{l_1, l_2}_{p, q+1}\left[\frac{-\mu y^{r_1} t}{(1-vx^{-m} t^{m_4})^{\beta}}\middle| \begin{matrix}(a_1, A_1), (a_2, A_2), \cdots, (a_p, A_p)\\ (b_1, 1), (b_2, B_2), \cdots, (b_{q+1}, B_{q+1})\end{matrix}\right].$$

Further the generalized polynomials $\overline{R}_n(x,y)$ will be denoted by $\overline{\overline{R}}_n(x,y)$ in case

$$(1.2) \qquad \alpha_1 = 0 = \nu_1.$$

On specializing the various parameters involved therein some interesting and new results for the classical polynomials are obtained.

The left hand side of (1.1) contain H-function defined by Fox [2].

The following notations have been used for brevity

(i) $(a_p) = a_1, a_2, \cdots, a_p$,

(ii) $[(a_p)]_n = \prod_{i=1}^{p} (a_i)_n = (a_1)_n (a_2)_n \cdots (a_p)_n$,

(iii) $\Delta_k[m; a] = \prod_{r=1}^{m}\left(\frac{a+r-1}{m}\right)_k = (a/m)_k \left(\frac{a+1}{m}\right)_k \cdots \left(\frac{a+m-1}{m}\right)_k$,

(iv) $$E = \frac{(-1)^{\sum_{j=1}^{l_1} B_j} \prod_{j=1}^{p} A_j^{A_j}}{(-1)^{\sum_{j=l_2+1}^{p} A_j} \prod_{j=2}^{r+1} B_j^{B_j}},$$

(v) $[(M_1(i,j))]_n=\prod_{j=1}^{l_2}\prod_{i=1}^{A_j}\left(\frac{i-a_j+A_jb_1}{A_j}\right)_n$,

(vi) $[1-(M_2(i,j))]_n=\prod_{j=l_2+1}^{p}\prod_{i=1}^{A_j}\left(1-\frac{a_j-A_jb_1+i-1}{A_j}\right)_n$,

(vii) $[(N_1(i,j))]_n=\prod_{j=l_1+1}^{q+1}\prod_{i=1}^{B_j}\left(\frac{i-b_j+B_jb_1}{B_j}\right)_n$,

(viii) $[1-(N_2(i,j))]_n=\prod_{j=2}^{l_1}\prod_{i=1}^{B_j}\left(1-\frac{B_j-B_jb_1+i-1}{B_j}\right)_n$,

(ix) $\Delta^1{}_k[m_4:1-(M_1(i,j))-n]=\prod_{j=1}^{l_2}\prod_{i=1}^{A_j}\prod_{l=1}^{m_4}\left(\frac{-(M_1(i,j))-n+l}{m_4}\right)_k$,

(x) $\Delta^2{}_k[m_4:(M_2(i,j))-n]=\prod_{j=l_2+1}^{p}\prod_{i=1}^{A_j}\prod_{l=1}^{m_4}\left(\frac{(M_2(i,j))-n+l-1}{m_4}\right)_k$,

(xi) $\Delta^3{}_k[m_4:1-(N_1(i,j))]=\prod_{j=l_1+1}^{q+1}\prod_{i=1}^{B_j}\prod_{l=1}^{m_4}\left(\frac{-(N_1(i,j))-n+l}{m_4}\right)_k$,

(xii) $\Delta^4{}_k[m_4:(N_2(i,j))-n]=\prod_{j=2}^{l_1}\prod_{i=1}^{B_j}\prod_{l=1}^{m_4}\left(\frac{(N_2(i,j))-n+l-1}{m_4}\right)_k$,

2. Differential equations for $\overline{\overline{R}}_n(x,y)$

(i) Case I ($m_4\beta=1$)

Consider

$$(2.1)\quad W_1=\left[\begin{array}{l}\Delta(m_4:-n),\ \Delta^3(m_4:1-(N_1(i,j))-n),\ \Delta^4(m_4:(N_2(i,j))-n),\\ \quad(1-\alpha-\beta b_1-n\beta):\\ \Delta^1(m_4:1-(M_1(i,j))-n),\ \Delta^2(m_4:(M_2(i,j))-n):\\ \quad Gx^{-m}\end{array}\right].$$

For the derivation of the differential equation we define an operator

$$\theta=\frac{m_4x}{m}\frac{\partial}{\partial x}.$$

Thus we have

$$(2.2)\quad \{\theta\Delta^1[m_4:1-(M_1(i,j))-n-m_4-\theta]\,\Delta^2[m_4:(M_2(i,j))-n-m_4-\theta]\}W_1$$

$$=\sum_{k=1}^{[n/m_4]}\frac{(-m_4k)\Delta_k[m_4:-n]\,\Delta^3{}_4[m_4:1-(N_1(i,j))-n]\,\Delta^4{}_k[m_4:(N_2(i,j))-n]\,G^k}{k!\,\Delta^1{}_k[m_4:1-(M_1(i,j))-n]\,\Delta^2{}_k[m_4:(M_2(i,j))-n]\,x^{mk}}$$

$$\times(1-\alpha-\beta b_1-n\beta)_k \Delta^1_k[m_4 : 1-(M_1(i,j))-n-m_4+m_4k]\Delta^2_k[m_4 : (M_2(i,j)) -n-m_4+m_4k]$$

$$= \sum_{k=1}^{[n/m_4]} \frac{(-m_4)\Delta_k[m_4 : -n]\Delta^3_k[m_4 : 1-(N_1(i,j))-n]\Delta^4_k[m_4 : (N_2(i,j))-n]G^{k+1}x^{-mk}}{(k-1)!\Delta^1_{k-1}[m_4 : 1-(M_1(i,j))-n]\Delta^2_{k-1}[m_4 : (M_2(i,j))-n] \times(1-\alpha-\beta b_1-n\beta)_k}.$$

Since $[m_4 : -n+m_4k]$ the product of the last m_4 factors in $\Delta_{k+1}[m_4 : -n]$ is zero in the interval $\left[\frac{n-m_4+1}{m_4}\right]\le k\le\left[\frac{n}{m_4}\right]$ in the R.H.S. we find

$$(2.3)\quad \{\theta\Delta^1[m_4 : 1-(M_1(i,j))-n-m_4-\theta]\ \Delta^2[m_4 : (M_2(i,j))-n-m_4-\theta]\}W_1$$

$$=-m_4G\sum_{k=0}^{[n/m_4]} \frac{\Delta_k[m_4 : -n]\ \Delta^3_k[m_4 : 1-(N_1(i,j))-n]\Delta^4_k[m_4 : (N_2(i,j))-n]}{k!\ \Delta^1_k\ [m_4 : 1-(M_1(i,j))-n]\ \Delta^2_k[m_4 : (M_2(i,j))-n]}$$

$$\times G^k x^{-mk-k}(1-\alpha-\beta b_1-n\beta)_k\ \Delta[m_4 :\ -n+m_4k]\Delta^3[m_4 : 1-(N_1(i,j))-n+m_4k]$$
$$\times\Delta^4[m_4 : (N_2(i,j))-n+m_4k]\ (1-\alpha-\beta b_1-n\beta+k).$$

Putting $G=\dfrac{(-v)(-m_4)m_4(\sum_{j=1}^{q+1}B_j+1-\sum_{j=1}^{p}A_j)}{(\mu Ey^{r_1})^{m_4}}$

Simplifying a little more, we achieve

$$(2.4)\quad [(M_1(i,j))+n+m_4+\theta-(m_4)]\ [n-(M_2(i,j))+m_4+\theta-(m_4)+1]$$
$$+\frac{m_4v}{(\mu Ey^{r_1})^{m_4}x^m}[\theta+n+1-(m_4)]\ [(N_1(i,j))+n+\theta-(m_4)]$$
$$\times[n-(N_2(i,j))-1-(m_4)]\left[-1+\alpha+\beta b_1+n\beta+\frac{\theta}{m_4}\right]\overline{R}_n(x,y)=0$$

which is one of the differential equations for the polynomial set.

(ii) Proceeding as above lines, we find another partial differential equation in terms of ϕ

$$(2.5)\quad \{(\phi-n)[(M_1(i,j))+m_4+\phi-(m_4)]\ [m_4-(M_2(i,j))+\phi-(m_4)+1]$$
$$+\frac{m_4v}{(\mu Ey^{r_1})^{m_4}x^m}[\phi+1-(m_4)]\ [(N_1(i,\ j))+\phi-(m_4)]\ [1-(N_2(i,j))+\phi-(m_4)]$$
$$\times\left[-1+\alpha+\beta b_1+\frac{\phi}{m_4}\right]\}\ \overline{R}_n(x,y)=0.$$

(iii) Proceeding exactly similar as in (1.2) we may deduce results for other three cases viz.

(2.6) $m_4\beta>1,\ m_4\beta<0,$ and $m_4\beta=0.$

3. Other solutions of the differential equations

(i) Case I $(m_4\beta=1)$

The complete solution of the differential equation (2.4) is given by

$$(3.1)\quad W=N_0 W_0+\sum_{g=1}^{m_4}\sum_{h=1}^{l_2} N_{g,h} W_{g,h}+\sum_{g=1}^{m_4}\sum_{u=l_2+1}^{p} N_{g,u} W_{g,u}$$

where N_0, $N_{g,h}$, $N_{g,u}$ are arbitrary parameters and ndeendent of x and $W_0=\bar{\bar{R}}_n(x,\ y)$

$$(3.2)\quad W_{g,h}=\sum_{i=1}^{A_h} X^{g-M_{1ih}-n-m_4}\ {}_{m_4\sum_{j=2}^{q+1}B_j+m_4+2}F_{n_4\sum_{j=1}^{p}A_j+1}[R_1]$$

where

$$[R_1]=\left[\begin{matrix}1,\ \Delta(m_4:M_{1ih}+m_4-g),\ \Delta^3(m_4:1-g-(N_1(i,j))+M_{1ih}+m_4),\\ \Delta^4(m_4:(N_2(i,j))-g+M_{1ih}+m_4),\ (2-\alpha+\beta b_1+\frac{M_{1ih}-g}{m_4});\\ \Delta^1(m_4:1-g-(M_1(i,j))+M_{1ih}+m_4),\ \Delta^2(m_4:(M_2(i,j))-g+M_{1ih}+m_4),\\ \left(\frac{M_{1ih}+n-g}{m_4}+2\right);\\ -v(\mu Ey^{r_1})^{-m_4}x^{-m}(-m_4)^{m_4w_1}\end{matrix}\right]$$

also

$$(3.3)\quad W_{g,u}=\sum_{i=1}^{A_u}X^{g-1-m_4-M_{2iu}-n}\ {}_{m_4\sum_{j=2}^{q+1}B_j+m_4+2}F_{m_4\sum_{j=1}^{p}A_j+1}[R_2]$$

$$[R_2]=\left[\begin{matrix}1,\ \Delta(m_4:1+m_4-g-M_{2iu}),\ \Delta^3(m_4:2-(N_1(i,j))-g-M_{2iu}+m_4),\\ \Delta^4(m_4:1+m_4-g-M_{2iu}+(N_2(i,j))),\ (2-\alpha+1\frac{-g-M_{2iu}-b_1}{m_4});\\ \Delta^1(m_4:2-(M_1(i,j))-g-M_{2iu}+m_4),\ \Delta^2(m_4:1+m_4-g-M_{2iu}\\ +(M_2(i,j)),\ \left(2+\frac{n+1-g-M_{2iu}}{m_4}\right);\\ -v(Ey^{r_1}\mu)^{-m_4}x^{-m}(-m_4)^{m_4w_1}\end{matrix}\right].$$

(ii) Similarly, the complete solution of the partial differential equation of

(2.5) is

(3.4) $W'=N'_0 W_0+\sum_{g=1}^{m_4}\sum_{h=1}^{l_2} N'_{g,h}W'_{g,h}+\sum_{g=1}^{m_4}\sum_{u=l_2+1}^{p} N'_{g,u} W'_{g,u}$

where N'_0, $N'_{g,h}$, $N'_{g,u}$ are arbitrary parameters and independent of x

(3.5) $$W'_{g,h}=\sum_{i=1}^{A_h} Y^{\frac{g-M_{1lh}-m_4}{m_4\sum_{j=2}^{q+1}B_j+m_4+2Fm_4\sum_{1}^{p}A_j+1}}[R_1]$$

(3.6) $$W'_{g,u}=\sum_{j=1}^{A_u} Y^{\frac{g-1-m_4+M_{2lu}}{m_4\sum_{j=2}^{q+1}B_j+m_4+2Fm_4\sum_{j=1}^{p}A_j+1}}[R_2]$$

where $[R_1]$ and $[R_2]$ are already defined.

(iii) Proceeding exactly as in (2.6) we achieve the (3.7) results for three cases viz. $m_4\beta>1$, $m_4\beta<0$ and $m_4\beta=0$.

Verification

We already know that $W_0=\overline{\overline{R}}_n(x,y)$ and satisfies (2.4), now, we shall show that $W_{g,h}$ and $W_{g,u}$ also satisfy (2.4).

The method of verification is similar to that of Rainville [1, p. 47]

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