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# ON A RELATION BETWEEN NÖRLUND SUMMABILITY AND LEBESGUE SUMMABILITY

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**1.1** Let $\Sigma a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$, $s_n=a_0+a_1+\cdots+a_n$. Let $\{p_n\}$ be a sequence of constants, real or complex such that

$$P_n=p_0+p_1+\cdots+p_n\neq 0,\ P_{-1}=p_{-1}=0$$

and let us write

$$(1.1.1)\qquad T_n=\sum_{\nu=0}^{n}p_{n-\nu}\,s_\nu\ ;\ t_n=\frac{T_n}{P_n}.$$

Then the series $\Sigma a_n$ is said to be *summable* $(N,p_n)$ *to sum* $s$, if $\lim_{n\to\infty} t_n$ exists and is equal to $s$ ([3], [9]).

It is to be observed ([2], [5]) that summability $(C,\alpha)$ and harmonic summability are special cases of $(N,p_n)$ summability, when $\{p_n\}$ is given by

$$(1.1.2)\qquad p_n=\binom{n+\alpha-1}{\alpha-1}=\frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)},\ (\alpha>-1)\ ;$$

and

$$(1.1.3)\qquad \begin{cases} p_n=1/(n+1) & (n\geq 0)\\ P_n=1/(1/2)+\cdots+(1/(n+1))\sim\log n, & \text{as } n\to\infty,\end{cases}$$

respectively.

The conditions for the regularity of the method of summability $(N,p_n)$ defined by (1.1.1), are:

$$(1.1.4)\qquad \lim_{n\to\infty}\frac{p_n}{P_n}=0,$$

and

$$(1.1.5)\qquad \sum_{i=0}^{n}|p_i|=O(P_n),\ \text{as } n\to\infty.$$

If $p_n$ is real, non-negative, and monotonic non-increasing the conditions of

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unless or otherwise stated $\Sigma$ denotes $\sum_0^\infty$

regularity (1.1.4) and (1.1.5) are automatically satisfied and the method $(N, p_n)$ is regular and hence harmonic summability is also regular. It is known that harmonic summability implies $(C, \alpha)$ summability for every $\alpha>0$.

The series $\Sigma a_n$ is said to be *summable by Lebesgue method (shortly (R, 1)-summable) to sum s,* if the sine series

(1.1.6) $$F(t)=\sum_{n=1}^{\infty} a_n\left(\frac{\sin nt}{n}\right)$$

is convergent in some interval $-\tau<t<\tau$, and if

(1.1.7) $$t^{-1}F(t)\to s, \text{ as } t\to 0, \qquad ([1]).$$

We write $H_n$ to denote the $n$-th harmonic sum of the sequence $\{s_n\}$.

**1.2** We set

(1.2.1) $(p_0+p_1x+\cdots+p_nx^n+\cdots)^{-1}=c_0+c_1x+\cdots+c_nx^n+\cdots(|x|<1; c_0=1)$

From (1.1.1), and (1.2.1), we obtain

(1.2.4) $$a_n=\sum_{\nu=0}^{n}c_{n-\nu}(T_\nu-T_{\nu-1})$$

from now onwards we take $a_0=0$, so that $T_0=0$.

**2.1** Concerning Lebesgue summability Szàsz ([6]) has proved the following result:

THEOREM A. *If $\Sigma a_n$ is summable $(C, 1-\alpha)$ for some positive $\alpha<1$, and if*

(2.1.1) $\sum_{\nu=1}^{n}|S_\nu^{-\alpha}|=O(n^{1-\alpha})$, *as $n\to\infty$*[1], *then the series $\Sigma a_n$ is summable by Lebesgue method.*

Recently Varshney ([8]) has proved an analogous theorem for harmonic summability. His result is as follows:

THEOREM B. *If a series $\Sigma a_n$ is harmonic summable and if*

(2.1.2) $\sum_{\nu=1}^{n}|H_\nu-H_{\nu-1}|=O(\log n)$, *as $n\to\infty$,*

*where $H_n=\sum_{\nu=1}^{n}(n-\nu+1)^{-1}s_\nu$, then $\Sigma a_n$ is summable by $(R, 1)$-method.*

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1) $S_n^{-\alpha}$ is the Cesaro sum of order $(-\alpha)$ of the series $\Sigma a_n$, i.e.

$S_n^{-\alpha}=\sum_{\nu=0}^{n}A_{n-\nu}a_\nu$, where $A_n^{-\alpha}=\binom{-\alpha+n}{n}$.

The object of this paper is to establish a couple of analogous theorems for Nörlund summability which covers both the Theorem $A$ and $B$ as special cases.

**2.2** Our main theorem is:

THEOREM 1. *If $\Sigma a_n$ is $(N,p_n)$-summable and, if*

(2.2.1) $$\sigma_n=\sum_{\nu=1}^{n}|T_\nu-T_{\nu-1}|=O(P_n)$$

*then $\Sigma a_n$ is $(R,1)$-summable, provided that $p_n$ is non-negative, non-increasing sequence such that $P_n\to\infty$, and*

(2.2.2) $$d_n=\sum_{\nu=0}^{n}c_\nu=O\left(\frac{1}{P_n}\right);$$

(2.2.3) $$\sum_{\nu=n+1}^{\infty}c_\nu=O\left(\frac{1}{P_n}\right),\ for\ n\geq 0;$$

(2.2.4) $$\sum_{\nu=n}^{\infty}\frac{P_{\nu-n}}{\nu(\nu+1)}=O\left(\frac{P_n}{n}\right),\ n\geq 1;$$

(2.2.5) $$\sum_{\nu=0}^{n}\frac{1}{P_\nu}=O\left(\frac{n}{P_n}\right);$$

*and*

(2.2.6) *for a positive integer $\mu$ and $n=[\mu t^{-1}]$, $\tau=[t^{-1}]$*
$$P_n=O(P_\mu P_\tau).$$

Combining Theorem 1 with Lemma 4 below, we also get the following interesting and simple result.

THEOREM 2. *Let $p_n$ be a positive, non-increasing sequence, such that $p_0=1$, $P_n\to\infty$, $\frac{p_{n+1}}{p_n}$ is non-decreasing sequence, and the condition (2.2.4) through (2.2.6) hold. If $\Sigma a_n$ is $(N,p_n)$-summable and if (2.2.1) holds, then $\Sigma a_n$ is also summable $(R,1)$.*

**2.3** We need the following lemmas for the proof of our theorems.

LEMMA 1. *If $\{p_n\}$ is a non-negative, non-increasing sequence such that the series $\sum_{\nu=n}^{\infty}P_{\nu-n}/\nu(\nu+1)$ converges, then $P_n/n\to 0$, as $n\to\infty$.*

PROOF. Since $p_n\geq 0$ and $n\,p_n\leq P_n$, we have

$$\frac{P_n}{n}-\frac{P_{n+1}}{n+1}=\frac{P_n-n\,p_{n+1}}{n(n+1)}\geq\frac{P_n-n\,p_n}{n(n+1)}\geq 0.$$

Obviously $\frac{P_n}{n}>0$. Thus the sequence $\frac{P_n}{n}$ is bounded and nonincreasing. Hence, there exists $\lim\limits_{n\to\infty}\frac{P_n}{n}=\alpha$, say. Then there exists an integer $N$ such that $\frac{P_n}{n}>\alpha/2\left(=\alpha-\frac{\alpha}{2}\right)$ for $n\geq N$. Hence, we have, for $n\geq N$,

$$\sum_{\nu=n}^{\infty}\frac{P_{\nu-n}}{\nu(\nu+1)}\geq\sum_{\nu=2n}^{\infty}\frac{P_{\nu-n}}{\nu-n}\,\frac{\nu-n}{\nu(\nu+1)}\geq\frac{\alpha}{2}\sum_{\nu=2n}^{\infty}\frac{\nu-n}{\nu(\nu+1)}=\infty,$$

which contradicts our assumption that the series $\sum\limits_{\nu=n}^{\infty}\frac{P_{\nu-n}}{\nu(\nu+1)}$ converges. Thus we see that $\alpha=0$ and hence the result.

LEMMA 2. *Let $\{p_n\}$ be a non-negative, non-increasing sequence such that*

$$\sum_{\nu=n}^{\infty}\frac{P_{\nu-n}}{\nu(\nu+1)}=O\left(\frac{P_n}{n}\right),$$

*then for $n\geq 1$,*

$$\sum_{\nu=n}^{\infty}\frac{P_\nu}{\nu(\nu+1)}=O\left(\frac{P_n}{n}\right). \tag{2.3.1}$$

PROOF. We have

$$\begin{aligned}\sum_{\nu=n}^{\infty}\frac{P_\nu}{\nu(\nu+1)}&=\sum_{\nu=n}^{\infty}\frac{(P_\nu-P_{\nu-n})}{\nu(\nu+1)}+\sum_{\nu=n}^{\infty}\frac{P_{\nu-n}}{\nu(\nu+1)}\\&=\sum_{\nu=n}^{2n}\frac{(P_\nu-P_{\nu-n})}{\nu(\nu+1)}+\sum_{\nu=2n+1}^{\infty}\frac{(P_\nu-P_{\nu-n})}{\nu(\nu+1)}+\sum_{\nu=n}^{\infty}\frac{P_{\nu-n}}{\nu(\nu+1)}\\&=O\left(\frac{P_{2n}}{n}\right)+\sum_{\nu=2n+1}^{\infty}\frac{1}{\nu(\nu+1)}\sum_{\mu=\nu-n+1}^{\nu}p_\mu+O\left(\frac{P_n}{n}\right)\\&=O\left(\frac{P_n}{n}\right)+O\left(\frac{P_n}{n}\right)+O\left(np_n\sum_{\nu=2n+1}^{\infty}\frac{1}{\nu(\nu+1)}\right)\\&=O\left(\frac{P_n}{n}\right),\qquad\text{by hypothesis, since } (n+1)\,p_n\leq P_n.\end{aligned}$$

LEMMA 3. *Let $p_n$ be non-negative non-increasing such that $\left\{\frac{P_n}{n}\right\}$ is a null sequence. If $\Sigma a_n$ is summable $(N,p_n)$, then*

(i) $$W_n=\sum_{\nu=n}^{\infty}\frac{T_\nu-T_{\nu-1}}{\nu}=o\left(\frac{P_n}{n}\right);$$

(ii) $$W_n'=\sum_{\nu=1}^{n}W_\nu=o(P_n).$$

PROOF. (i) We may assume, without any loss of generality, that $T_n=o(P_n)$. By Abel's transformation and by hypothesis, as $m\to\infty$, we have

$$\begin{aligned}W_n=\sum_{\nu=n}^{\infty}\frac{T_\nu-T_{\nu-1}}{\nu}&=\sum_{\nu=n}^{m-1}\frac{1}{\nu(\nu+1)}\sum_{\mu=0}^{\nu}(T_\mu-T_{\mu-1})\\&\quad+\frac{1}{m}\sum_{\mu=0}^{m}(T_\mu-T_{\mu-1})-\frac{1}{n}\sum_{\mu=0}^{n-1}(T_\mu-T_{\mu-1})\\&=\sum_{\nu=n}^{m-1}\frac{T_\nu}{\nu(\nu+1)}+\frac{T_m}{m}-\frac{T_{n-1}}{n}\\&=o\left(\sum_{\nu=n}^{\infty}\frac{P_\nu}{\nu(\nu+1)}\right)+o\left(\frac{P_{n-1}}{n}\right)\\&=o\left(\frac{P_n}{n}\right)+o\left(\frac{P_{n-1}}{n}\right)\qquad\text{(by Lemma 2).}\\&=o\left(\frac{P_n}{n}\right),\quad\text{by regularity of the method } (N,\ p_n)\end{aligned}$$

(ii) $$\begin{aligned}W_n'=\sum_{\nu=1}^{n}W_\nu&=\sum_{\nu=1}^{n}\nu\Delta W_\nu+nW_{n+1}\\&=\sum_{\nu=1}^{n}\nu\frac{(T_\nu-T_{\nu-1})}{\nu}+nW_{n+1}\\&=\sum_{\nu=1}^{n}(T_\nu-T_{\nu-1})+nW_{n+1}\\&=T_n+nW_{n+1}\qquad\text{(since } T_0=p_0\,a_0=0)\\&=o(P_n)+o\left(n\frac{P_{n+1}}{n+1}\right)\\&=o(P_n),\end{aligned}$$

by hypothesis and (i). Hence the result.

LEMMA 4. ([7], Lemma 2). *If $\{p_n\}$ is a positive and non-increasing sequence such that $p_0=1$, $P_n\to\infty$, and $\{p_{n+1}/p_n\}$ is a non-decreasing sequence then, for $n\geq 0$,*

$$d_n=\sum_{\nu=n+1}^{\infty}|c_\nu|=\sum_{\nu=0}^{n}c_\nu=O(1/P_n).$$

REMARK. The identity

$$d_n=\sum_{\nu=n+1}^{\infty}|c_\nu|=\sum_{\nu=0}^{n}c_\nu$$

is obtained by virtue of Kaluza's result. (see [2], Theorem 22).

LEMMA 5. ([4], Lemma 3). *Let $\Delta_n^m\ \phi(nt)$ denote the m-th difference of $\phi(nt)$ with respect to n. Then we have*

(2.3.2) $$\Delta_n^m\ \phi(nt)=O(t^{m-p}/n^p),$$

*where m is a non-negative integer and* $\phi(t)=(\sin t/t)^p$.

LEMMA 6. *If $p_n$ is such that it satisfies all the conditions of the theorem except* (2.2.3), *then the series*

(2.3.3) $$\sum_{n=0}^{\infty}c_n\frac{\sin(n+\nu)t}{(n+\nu)t}=s_\nu(t)$$

*is absolutely convergent and for* $m=0,\ 1,\ 2,\cdots\cdots$

(2.3.4) $$\Delta_\nu^m\ S_\nu(t)=O\Big(\frac{t^{m-1}}{\nu P_\tau}\Big).$$

PROOF. Absolute convergence of the series (2.3.3) follows from the hypotheses, since $\sum_{n=1}^{\infty}|c_n|<\infty$. To prove (2.3.4) we have by setting $\phi(t)=(\sin\ t/t)$

$$\begin{aligned}\Delta_\nu^m\ S_\nu(t)&=\Delta_\nu^m\Big\{\sum_{n=0}^{\infty}c_n\phi((n+\nu)t\Big\}=\sum_{n=0}^{\infty}c_n\ \Delta_\nu^m\ \phi((n+\nu)t)\\&=\Big(\sum_{n=0}^{\tau}+\sum_{n=\tau+1}^{\infty}\Big)c_n\ \Delta_\nu^m\phi((\mathrm{n}+\nu)t)\\&=S_\nu^{(1)}(t)+S_\nu^{(2)}(t),\ \text{say.}\end{aligned}$$

Now, by (2.2.3) and Lemma 5, we have

$$\begin{aligned}S_\nu^{(2)}(\mathrm{t})&=\sum_{n=\tau+1}^{\infty}c_n\Delta_\nu^m\phi((n+\nu)t)=O\Big(\sum_{n=\tau+1}^{\infty}|c_n|.\ \frac{t^{m-1}}{(n+\nu)}\Big)\\&=O\Big(\frac{t^{m-1}}{\nu+\tau+1}\sum_{n=\tau+1}^{\infty}|c_n|\Big)=O\Big(\frac{t^{m-1}}{\nu P_\tau}\Big).\end{aligned}$$

And, on applying Abel's transformation to the expression in $S_\nu^{(1)}(t)$, we obtain

$$S_\nu^{(1)}(t)=\sum_{n=0}^{\tau-1} d_n\ \Delta_n\Big\{\Delta_\nu^m\ \phi((n+\nu)t)\Big\}+d_\tau\ \Delta_\nu^m \phi((\tau+\nu)t)$$

$$=\sum_{n=0}^{\tau-1} d_n\ \Delta_n^{m+1}(\phi(n+\nu)t)+d_\tau\ \Delta_\nu^m\phi((\tau+\nu)t)$$

$$=O\Big(\sum_{n=0}^{\tau-1}\frac{1}{P_n}\ \frac{t^m}{(n+\nu)}\Big)+O\Big(\frac{1}{P_\tau}\ \frac{t^{m-1}}{(\nu+\tau)}\Big)$$

$$=O\Big(\frac{t^m}{\nu}\cdot\sum_{n=0}^{\tau-1}\frac{1}{P_n}\Big)+O\Big(\frac{t^{m-1}}{\nu P_\tau}\Big)$$

$$=O\Big(\frac{t^{m-1}}{\nu P_\tau}\Big),\ \text{by (2.2.5)}.$$

This completes the proof of lemma 6.

2.4. PROOF OF THEOREM 1. We may assume without any loss of generality that $T_n=o(P_n)$, as $n\to\infty$. By (1.1.1) and (1.2.4) we have

$$F(t)=\sum_{n=1}^{\infty} n^{-1}\sin nt\ \sum_{\nu=1}^{n} c_{n-\nu}(T_\nu-T_{\nu-1})$$

$$=t\sum_{\nu=1}^{\infty}(T_\nu-T_{\nu-1})\ \sum_{n=\nu}^{\infty} c_{n-\nu}\frac{\sin nt}{nt},$$

the interchange of order of summations being legitimate, since by the following considerations the double series is absolutely convergent.

Since, by hypothesis $\Sigma|c_n|<\infty$, we have

$$\Big|\sum_{n=\nu}^{\infty} n^{-1}\ c_{n-\nu}\ \sin nt\Big|\le\frac{1}{\nu}\sum_{n=0}^{\infty}|c_n|=O(1/\nu),$$

and hence, as $m\to\infty$,

$$\sum_{\nu=1}^{m}\Big|(T_\nu-T_{\nu-1})\Big|\sum_{n=\nu}^{\infty} c_{n-\nu}\frac{\sin nt}{nt}=O(1)\sum_{\nu=1}^{m}\frac{|T_\nu-T_{\nu-1}|}{\nu}$$

$$=O(1)(m^{-1}\ \sigma_m)+O(1)\Big(\sum_{\nu=1}^{m-1}\frac{P_\nu}{\nu(\nu+1)}\Big)$$

$$=O(1)\frac{P_m}{m}+O(1)\Big(\sum_{\nu=1}^{m-1}\frac{P_\nu}{\nu(\nu+1)}\Big)$$

$$=O(1),\qquad \text{by Lemmas 1 and 2.}$$

Thus

$$\frac{F(t)}{t}=\sum_{\nu=1}^{\infty}(T_\nu-T_{\nu-1})\sum_{n=\nu}^{\infty} n^{-1}\ c_{n-\nu}\frac{\sin nt}{t}.$$

$$=\sum_{\nu=1}^{\infty}(T_\nu-T_{\nu-1})S_\nu(t)$$

$$=\Big(\sum_{\nu=1}^{n}+\sum_{\nu=n+1}^{\infty}\Big)(T_\nu-T_{\nu-1})S_\nu(t)$$

$$=\Sigma_1+\Sigma_2,\ \text{say.} \tag{2.4.1}$$

Now,

$$|\Sigma_2|=\Big|\sum_{\nu=n+1}^{\infty}(T_\nu-T_{\nu-1})\ S_\nu(t)\Big|$$

$$=O\Big(\sum_{\nu=n+1}^{\infty}|(T_\nu-T_{\nu-1})|\frac{1}{\nu t\ P_\tau}\Big)$$

$$=O\Big[\frac{1}{tP_\tau}\Big(\sum_{\nu=n+1}^{\infty}\frac{\sigma_\nu}{\nu(\nu+1)}-\frac{\sigma_n}{n+1}\Big)\Big]$$

$$=O\Big[\frac{\tau}{tP_\tau}\Big(\sum_{\nu=n+1}^{\infty}\frac{P_\nu}{\nu(\nu+1)}+\frac{P_n}{n+1}\Big)\Big]$$

$$=O\Big[\frac{\tau}{P_\tau}\ \frac{P_n}{n}\Big]=O\Big(\frac{P_\mu}{\mu}\Big)$$

$$=O(1)\cdot\frac{P_\mu}{\mu},\ \text{by (2.2.6) and Lemmas 2 and 6.} \tag{2.4.2}$$

Next, we have

$$\Sigma_1=\sum_{\nu=1}^{n}(T_\nu-T_{\nu-1})S_\nu(t)$$

$$=\sum_{\nu=1}^{n}(W_\nu-W_{\nu+1})\nu S_\nu(t)$$

$$=\sum_{\nu=1}^{n}W_\nu[\nu S_\nu(t)-(\nu-1)S_{\nu-1}(t)]-nW_{n+1}\ S_n(t)$$

$$=-\sum_{\nu=1}^{n}W_\nu\,\nu[S_{\nu-1}(t)-S_\nu(t)]+\sum_{\nu=1}^{n}W_\nu\ S_{\nu-1}(t)-nW_{n+1}\ S_n(t)$$

$$=-\Sigma_{1,1}+\Sigma_{1,2}-nW_{n+1}\ S_n(t)$$

where, by Lemma 1, 3(ii) and 6,

$$\Sigma_{1,1}=\sum_{\nu=1}^{n}\nu W_\nu\Delta S_{\nu-1}(t)$$

$$=\sum_{\nu=1}^{n}\Big\{\sum_{\mu=1}\mu W_\mu\Big\}\Delta^2 S_{\nu-1}(t)+\Delta S_n(t)\sum_{\nu=1}^{n}\nu W_\nu$$

$$=o\Big(\sum_{\nu=1}^{n}\nu P_\nu \frac{t}{\nu \nu P_\tau}\Big)+o\Big(\frac{1}{nP_\tau}\cdot nP_n\Big)$$

$$=o\Big(nt\frac{P_n}{P_\tau}\Big)+o\Big(\frac{P_n}{P_\tau}\Big)$$

$$=o(\mu P_\mu)+o(P_\mu)$$

$$=o(1)$$

since $\sum_{\nu=1}^{n}\nu W_\nu=o\Big(\sum_{\nu=1}^{n}\nu\frac{P_\nu}{\nu}\Big)=o(nP_n)$, and by applying Abel's transformation twice, writing $W'_m=\sum_{\mu=1}^{m}W_\mu$, and by virtue of Lemmas 1, 3(ii) and 6, we have

$$\Sigma_{1,2}\equiv\sum_{\nu=1}^{n}\Big(\sum_{m=1}W'_m\Big)\Delta^2 S_{\nu-1}(t)+\Delta S_n(t)\sum_{\nu=1}^{n}W_\nu'+S_n(t)\;W_n'$$

$$=o\Big(\sum_{\nu=1}^{n}\nu P_\nu\frac{t}{\nu\nu P_\tau}\Big)+o\Big(\frac{1}{nP_\tau}\sum_{\nu=1}^{n}P_\nu\Big)+o\Big(\frac{P_n}{nt\,P_\tau}\Big)$$

(by Lemmas 3(ii) and 6)

$$=o\Big(\frac{nt\,P_n}{P_\tau}\Big)+o\Big(\frac{P_n}{P_\tau}\Big)+o\Big(\frac{P_\mu}{\mu}\Big)=o(1)$$

and, by Lemmas 1, 3(i) and 6, we have $nW_{n+1}\;S_n(t)=o(1)\frac{P_\mu}{\mu}=o(1)$. Hence

(2.4.3) $$\Sigma_1=o(1).$$

Therefore, from (2.4.1), (2.4.2) and (2.4.3), we obtain

$$t^{-1}F(t)=O(1)\;\frac{P_\mu}{\mu}+o(1),\ \text{as } t\to 0.$$

Consequently,

$$\limsup_{t\to+0}\; t^{-1}|F(t)|\le o(1)\frac{P_\mu}{\mu}$$

being arbitrary large and $O(1)$ independent of $\mu$, we get finally

$$t^{-1}F(t)\to 0 \text{ as } t\to 0.$$

This terminates the proof of theorem 1.

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