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ON FINITE SUMMATION FORMULAE FOR THE H-FUNCTION OF TWO VARIABLES

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0. Abstract

In the present paper, we obtain two new and interesting finite summation formulae for the H-function of two variables in a very neat and elegant form. The novel feature of the paper is that the method used here in deriving these formulae is simple and direct and does not impose heavy restrictions on the parameters involved. On account of the most general nature of the H-function of two variables, a number of related finite summation formulae for a number of other useful functions can also be obtained as special cases of our results. As an illustration, we have obtained here from our main results, the corresponding finite summation formulae for Kampé de Fériet function, Appell's function and Gauss' hypergeometric function which are also believed to be new.

1. Introduction

The parameters of the H-function of two variables [6] occurring in the present paper are displayed in the following contracted notation [5]:

$$H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\left|\begin{matrix}(a_j:\alpha_j,A_j)_{1,p_1}:(c_j,\ r_j)_{1,p_2}:(e_j,\ E_j)_{1,p_3}\\(b_j:\beta_j,B_j)_{1,q_1}:(d_j,\ \delta_j)_{1,q_2}:(f_j,\ F_j)_{1,q_3}\end{matrix}\right.\right]$$

$$=(2\pi i)^{-2}\int_{L_1}\int_{L_2}\phi(s,t)\theta_1(s)\theta_2(t)x^s y^t\,ds\,dt. \tag{1.1}$$

where, for convenience, let $(a_j:\alpha_j,\ A_j)_{n_1+1,p_1}$ and $(c_j,\ r_j)_{n_2+1,p_2}$ abbreviate the parameters sequence $(a_{n_1+1}:\alpha_{n_1+1},\ A_{n_1+1}),\cdots,\ (a_{p_1}:\alpha_{p_1},\ A_{p_1})$ and $(c_{n_2+1},\ r_{n_2+1}),\cdots,\ (c_{p_2},\ r_{p_2})$ respectively. For the integers n_i and p_i such that $0\leq n_i\leq p_i$ $(i=1,2)$ and similar interpretations for $(b_j:\beta_j,B_j)_{m_1+1,q_1}$, $(d_j,\ \delta_j)_{m_2+1,q_2}$ and so on.

Also, $\phi(s,t)=\prod_1^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)\Big[\prod_{n_1+1}^{p_1}\Gamma(a_j-\alpha_j s-A_j t)\times$

$\Gamma(1-b_j+\beta_j s+B_j t\Big]^{-1}$

$$\theta_1(s)=\prod_1^{n_2}\Gamma(1-c_j+r_js)\prod_1^{m_2}\Gamma(d_j-\delta_js)\Big[\prod_{n_2+1}^{p_2}\Gamma(c_j-r_js)\times\Gamma(1-d_j+\delta_js)\Big]^{-1}$$

and with $\theta_2(t)$ defined analogously in terms of the parameter sets $(e_j, E_j)_{1,p_3}$, $(f_j, F_j)_{1,q_3}$.

The conditions on parameters of the H-function of two variables, its asymptotic expansions, some of its properties, particular cases, nature of contours L_1 and L_2 in (1.1) etc. can be referred to in a paper by Mittal and Gupta [6].

2. Simplifying notations

Since, only the parameters subscripted 1 in the definition of the H-function of two variables undergo change in our main results, therefore, to simplify notational problems, we specify only these parameters in our main finite series. Thus,

$H[a_1+1-r : \alpha_1 h, \alpha_1 k), (b_1+1 : \beta_1 h, \beta_1 k)]$ would represent the H-function of two variables defined by (1.1), but having a_1 replaced by a_1+1-r, α_1 replaced by $\alpha_1 h$, A_1 replaced by $\alpha_1 k$, b_1 replaced by b_1+1, β_1 replaced by $\beta_1 h$, B_1 replaced by $\beta_1 k$ and rest of the parameters being same as in (1.1).

Main results

$$\frac{\alpha_1}{\beta_1}\sum_{r=1}^{n}\frac{1}{\Gamma(1-a_1+r+b_1\alpha_1/\beta_1)}H[(a_1+1-r ;\ \alpha_1 h, \alpha_1 k),\ (b_1+1 : \beta_1 h, \beta_1 k)]$$

$$=\frac{1}{\Gamma(1-a_1+n+b_1\alpha_1/\beta_1)}H[(a_1-n : \alpha_1 h, \alpha_1 k),\ (b_1 : \beta_1 h, \beta_1 k)]$$

$$-\frac{1}{\Gamma(1-a_1+b_1\alpha_1/\beta_1)}H[(a_1 : \alpha_1 h, \alpha_1 k), (b_1 : \beta_1 h, \beta_1 k)] \tag{2.1}$$

$$\beta_1/\alpha_1\sum_{r=1}^{n}\frac{(-1)^r}{\Gamma((1-a_1)\beta_1/\alpha_1+b_1+r)}H[(a_1-1 : \alpha_1 h,\ \alpha_1 k),\ (b_1+r-1 : \beta_1 h,\ \beta_1 k)]$$

$$=\frac{(-1)^n}{\Gamma((1-a_1)\beta_1/\alpha_1+b_1+n)}H[(a_1 : \alpha_1 h, \alpha_1 k),\ (b_1+n : \beta_1 h, \beta_1 k)]$$

$$-\frac{1}{\Gamma((1-a_1)\beta_1/\alpha_1+b_1)}=H[(a_1 : \alpha_1 h, \alpha_1 k),\ (b_1 : \beta_1 h, \beta_1 k)] \tag{2.2}$$

PROOFS of (2.1) and (2.2).

To prove (2.1), we first give the following simple three term contiguous relation for the H-function of two variables:

$$\alpha_1/\beta_1 \frac{1}{\Gamma(1-a_1+r+b_1\alpha_1/\beta_1)} H[(a_1+1-r : \alpha_1 h, \alpha_1 k),\ (b_1+1 : \beta_1 h, \beta_1 k)]$$

$$= \frac{1}{\Gamma(1-a_1+r+b_1\alpha_1/\beta_1)} H[(a_1-r : \alpha_1 h, \alpha_1 k),\ (b_1 : \beta_1 h, \beta_1 k)]$$

$$- \frac{1}{\Gamma(-a_1+r+b_1\alpha_1/\beta_1)} H[(a_1-r+1 : \alpha_1 h, \alpha_1 k),\ (b_1 : \beta_1 h, \beta_1 k)] \tag{2.3}$$

To prove (2.3), we note from the definition of the H-function of two variables that the replacement of (a_1-r) by (a_1-r+1) in (1.1) and the application of the recurrence formula $\Gamma(z+1)=z\Gamma(z)$ is equivalent to the introduction of the additional multiplying factor $(-a_1+r+\alpha_1 hs+\alpha_1 kt)^{-1}$ into the contour integral fermat for the H-function of two variables. Similarly, the replacement of b_1 by b_1+1 will introduce an additional multiplying factor $(-b_1+\beta_1 hs+\beta_1 kt)$. Consequently, we can simply form a following 3-term recurrence relation involving undetermined coefficients A, B and C:

$$AH[(a_1+1-r;\ \alpha_1 h, \alpha_1 k),\ (b_1+1 : \beta_1 h, \beta_1 k)]$$

$$=BH[(a_1-r : \alpha_1 h, \alpha_1 k),\ (b_1 : \beta_1 h, \beta_1 k)]$$

$$+CH[(a_1-r+1 : \alpha_1 h, \alpha_1 k),\ (b_1 : \beta_1 h, \beta_1 k)] \tag{2.4}$$

and then require that

$A(-b_1+\beta_1 hs+\beta_1 kt)=B(-a_1+r+\alpha_1 hs+\alpha_1 kt)+C$, by an identity in s and t. Hence, A, B and C can be evaluated. On evaluating the values of these quantities, and after a little simplification, we easily arrive at the required result (2.3). Now, if we put $r=1, 2, 3, \cdots, n$ in (2.3), and take the sum, we observe that the resulting series on the right hand side collapses, and after a little simplification, we get, the required result (2.1).

To prove (2.2), we start with the following contiguous relation:

$$\frac{(-1)^r}{\Gamma((1-a_1)\beta_1/\alpha_1+b_1+r)} H[(a_1-1, \alpha_1 h, \alpha_1 k),\ (b_1+r-1 : \beta_1 h, \beta_1 k)]$$

$$= \frac{(-1)^r}{\Gamma(1-a_1)\beta_1/\alpha_1+b_1+r)} H[(a_1 : \alpha_1 h, \alpha_1 k),\ (b_1+r : \beta_1 h, \beta_1 k)]$$

$$+ \frac{(-1)^r}{\Gamma((1-a_1)\beta_1/\alpha_1+b_1+r-1)} H[(a_1 : \alpha_1 h, \alpha_1 k),\ (b_1+r-1 : \beta_1 h, \beta_1 k)] \tag{2.5}$$

which can also be proved as in the case of (2.3), and then proceed in a manner similar to that given during the proof of (2.1).

3. Special cases

If we specialize the parameters of the various H-functions of two variables involved in the series (2.1) and (2.2), such that all of them reduce to Kampé de Fériet functions [1], we get by virtue of a known formula [4] after a little simplification, the following interesting series involving Kampé de Fériet functions.

$$(b_1-1)\sum_{r=1}^{n}\frac{\Gamma(a_1+r-1)}{\Gamma(1+a_1-b_1+r)}F^{p_1,\,p_2}_{q_1,\,q_2}\left[\begin{matrix}a_1+r-1,\,(a_j)_{2,\,p_1}:(c_j)_{1,\,p_2};\,(e_j)_{1,\,p_2}\\ b_1-1,\quad (b_j)_{2,\,q_1}:(d_j)_{1,\,q_2};\,(f_j)_{1,\,q_2}\end{matrix}\middle|\,x,y\right]$$
$$=\frac{\Gamma(a_1+n)}{\Gamma(1+a_1-b_1+n)}F^{p_1,\,p_2}_{q_1,\,q_2}\left[\begin{matrix}a_1+n,\,(a_j)_{2,\,p_1}:(c_j)_{1,\,p_2};\,(e_j)_{1,\,p_2}\\ (b_j)_{1,\,q_1}:(d_j)_{1,\,q_2};\,(f_j)_{1,\,q_2}\end{matrix}\middle|\,x,y\right]$$
$$-\frac{\Gamma(a_1)}{\Gamma(1+a_1-b_1)}F^{p_1,\,p_2}_{q_1,\,q_2}\left[\begin{matrix}(a_j)_{1,\,p_1}:(c_j)_{1,\,p_2};\,(e_j)_{1,\,p_2}\\ (b_j)_{1,\,q_1}:(d_j)_{1,\,q_2};\,(f_j)_{1,\,q_2}\end{matrix}\middle|\,x,\,y\right] \tag{3.1}$$

$$a_1\sum_{r=1}^{n}\frac{(-1)^r}{\Gamma(1+a_1-b_1+r)\,\Gamma(1+b_1-r)}$$
$$F^{p_1,\,p_2}_{q_1,\,q_2}\left[\begin{matrix}a_1+1,\,(a_j)_{2,\,p_1}:(c_j)_{1,\,p_2};\,(e_j)_{1,\,p_2}\\ 1+b_1-r,\,(b_j)_{2,\,q_1}:(d_j)_{1,\,q_2};\,(f_j)_{1,\,q_2}\end{matrix}\middle|\,x,\,y\right]$$
$$=\frac{(-1)^n}{\Gamma(1+a_1-b_1+n)\Gamma(b_1-n)}F^{p_1,\,p_2}_{q_1,\,q_2}\left[\begin{matrix}(a_j)_{1,\,p_1}:(c_j)_{1,\,p_2};\,(e_j)_{1,\,p_2}\\ (b_1-n),\,(b_j)_{2,\,q_1}:(d_j)_{1,\,q_2};\,(f_j)_{1,\,q_2}\end{matrix}\middle|\,x,y\right]$$
$$-\frac{1}{\Gamma(1+a_1-b_1)\Gamma(b_1)}F^{p_1,\,p_2}_{q_1,\,q_2}\left[\begin{matrix}(a_j)_{1,\,p_1}:(c_j)_{1,\,p_2};\,(e_j)_{1,\,p_2}\\ (b_j)_{1,\,q_1}:(d_j)_{1,\,q_2};\,(f_j)_{1,\,q_2}\end{matrix}\middle|\,x,\,y\right] \tag{3.2}$$

Also, if we put $p_1=q_1=p_2=1$ and $q_2=0$ in the equations (3.1) and (3.2), we get, after a little simplification, the following series involving Appell's functions respectively:

$$(b-1)\sum_{r=1}^{n}\frac{\Gamma(a+r-1)}{\Gamma(1+a-b+r)}F_1(a+r-1:\alpha,\beta:b-1:x,y)$$
$$=\frac{\Gamma(a+n)}{\Gamma(1+a-b+n)}F_1(a+n:\alpha,\ \beta:b:x,\ y)$$
$$-\frac{\Gamma(a)}{\Gamma(1+a-b)}F_1(a:\alpha,\ \beta:b:x,\ y) \tag{3.3}$$

$$a\sum_{r=1}^{n}\frac{(-1)^r}{\Gamma(1+a-b+r)\Gamma(1+b-r)}F_1(a+1:\alpha,\ \beta:1+b-r:x,\ y)$$
$$=\frac{(-1)^n}{\Gamma(1+a-b+n)\Gamma(b-n)}F_1(a:\alpha,\ \beta:b-n:x,\ y)$$

$$-\frac{1}{\Gamma(1+a-b)\,\Gamma(b)}F_1(a:\alpha,\ \beta:b:x,\ y) \tag{3.4}$$

Again, if in the equations (3.3) and (3.4), we put $y=x$ and use a known result [2], we get, after a little simplification, the following interesting series involving Gauss' hypergeometric functions [3] respectively:

$$(b-1)\sum_{r=1}^{n}\frac{\Gamma(a+r-1)}{\Gamma(1+a-b+r)}\,{}_2F_1\left(\begin{matrix}a+r-1,\ \alpha\\ b-1\end{matrix};x\right)$$

$$=\frac{\Gamma(a+n)}{\Gamma(1+a-b+n)}\,{}_2F_1\left(\begin{matrix}a+n,\ \alpha\\ b\end{matrix};x\right)-\frac{\Gamma(a)}{\Gamma(1+a-b)}\,{}_2F_1\left(\begin{matrix}a,\ \alpha\\ b\end{matrix};x\right) \tag{3.5}$$

$$a\sum_{r=1}^{n}\frac{(-1)^r}{\Gamma(1+a-b+r)\Gamma(1+b-r)}\,{}_2F_1\left(\begin{matrix}1+a,\ \alpha\\ 1+b-r\end{matrix};x\right)$$

$$=\frac{(-1)^n}{\Gamma(1+a-b+n)\Gamma(b-n)}\,{}_2F_1\left(\begin{matrix}a,\ \alpha\\ b-n\end{matrix};x\right)$$

$$-\frac{1}{\Gamma(1+a-b)\Gamma(b)}\,{}_2F_1\left(\begin{matrix}a,\ \alpha\\ b\end{matrix};x\right) \tag{3.6}$$

Related finite series for other special functions can also be obtained from (2.1) and (2.2) by reducing the H-function of two variables into some other simple functions.

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