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A NOTE ON GENERALIZED LAGUERRE POLYNOMIALS

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Recently, R.B. Sahu [3] has generalized the Laguerre polynomials by means of the following generating relation,

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x,k)t^n=(1-t)^{-\alpha-1}\exp\left[\frac{-x^k t}{(1-t)^k}\right] \tag{1}$$

$L_n^{(\alpha)}(x,k)$ being a polynomial of degree n in x^k. In the present investigation, we shall prove the following relation for generalized Laguerre polynomials $L_n^{(\alpha)}(x,k)$ defined by (1),

$$\sum_{n=0}^{\infty}(-1)^n L_n^{(1-\alpha-2n)}(x,k)t^n=(1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{\alpha-2}\exp\left[\frac{4x^k t(1+\sqrt{1-4t})^k}{2^k(1+\sqrt{1-4t})^2}\right] \tag{2}$$

PROOF. Since,

$$(1-t)^{-\alpha-1}\exp\left(\frac{-x^k t}{(1-t)^k}\right)=(1-t)^{-\alpha}(1-t)^{-1}\exp\left(\frac{-x^k t}{(1-t)^k}\right)$$

and $\exp\left(\frac{-x^k t}{(1-t)^k}\right)$ is independent of the parameter α, it follows that

$$L_n^{(\alpha)}(x,k)=\sum_{\nu=0}^{n}\frac{(\alpha)_\nu}{\nu!}L_{n-\nu}(x,k).$$

Replacing α by $(1-\alpha-2n)$, we arrive at

$$L_n^{(1-\alpha-2n)}(x,k)=\sum_{\nu=0}^{n}\frac{(\alpha)_{2n}(-1)^\nu L_{n-\nu}(x,k)}{(\alpha)_{2n-\nu}\,\nu!}$$

Now consider,

$$\sum_{n=0}^{\infty}(-1)^n L_n^{(1-\alpha-2n)}(x,k)t^n=\sum_{n=0}^{\infty}\sum_{\nu=0}^{n}\frac{(\alpha)_{2n}(-1)^{n-\nu}}{(\alpha)_{2n-\nu}\,\nu!}L_{n-\nu}(x,k)t^n$$

$$=\sum_{n,\nu=0}^{\infty}\frac{(\alpha)_{2n+2\nu}(-1)^n}{(\alpha)_{2n+\nu}\,\nu!}L_n(x,k)t^{n+\nu} \tag{3}$$

But

$$\frac{(\alpha)_{2n+2\nu}}{(\alpha)_{2n+\nu}}=\frac{\left(\frac{\alpha}{2}+n\right)_\nu\left(\frac{\alpha}{2}+n+\frac{1}{2}\right)_\nu 4^\nu}{(\alpha+2n)_\nu},$$

we have from equation (3)

$$\sum_{n=0}^{\infty}(-1)^n L_n^{(1-\alpha-2n)}(x,k)t^n=\sum_{n,\,\nu=0}^{\infty}\frac{\left(\frac{\alpha}{2}+n\right)_\nu\left(\frac{\alpha}{2}+n+\frac{1}{2}\right)_\nu(4t)^\nu}{(\alpha+2n)_\nu}L_n(x,k)(-t)^n$$

$$=\sum_{n=0}^{\infty}{}_2F_1\left[\begin{matrix}\frac{\alpha}{2}+n,\ \frac{\alpha}{2}+n+\frac{1}{2}\ ; \\ \alpha+2n\ ;\end{matrix}\ 4t\right]L_n(x,k)(-t)^n.$$

Letting $c=\frac{\alpha}{2}+n,\ u=4t$ in the identity [1]

$${}_2F_1\left[\begin{matrix}c,\ c+\frac{1}{2}\ ; \\ 2c\ ;\end{matrix}\ u\right]=\frac{1}{\sqrt{(1-u)}}\left(\frac{2}{1+\sqrt{(1-u)}}\right),$$

we get

$$\sum_{n=0}^{\infty}(-1)^n L_n^{(1-\alpha-2n)}(x,k)t^n=\frac{1}{\sqrt{1-4t}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{\alpha-1}$$

$$\times\sum_{n=0}^{\infty}L_n(x,k)\left(\frac{-4t}{(1+\sqrt{1-4t})^2}\right)^n.$$

Using the fact that,

$$\sum_{n=0}^{\infty}L_n(x,k)t^n=(1-t)^{-1}\exp\left(\frac{-x^k t}{(1-t)^k}\right),$$

we finally obtain,

$$\sum_{n=0}^{\infty}(-1)^n L_n^{(1-\alpha-2n)}(x,k)t^n=(1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{\alpha-2}$$

$$\times\exp\left[\frac{4x^k t(1+\sqrt{1-4t})^k}{2^k(1+\sqrt{1-4t})^2}\right].$$

Putting $k=1$ and remembering that $L_n^{(\alpha)}(x,1)=L_n^{(\alpha)}(x)$, we have from equation (2)

$$\sum_{n=0}^{\infty}(-1)^n L_n^{(1-\alpha-2n)}(x)t^n=(1-4t)^{-\frac{1}{2}}\left(\frac{2}{1+\sqrt{1-4t}}\right)^{\alpha-2}\times\exp\left[\frac{2xt}{(1+\sqrt{1-4t})}\right],$$

a similar relation for Laguerre polynomials $L_n^{(\alpha)}(x)$.

(ii) Next, we shall show that

$$L_n^{(\alpha)}(x,k)=\sum_{\nu=0}^{n}\frac{(\alpha-\beta)_\nu}{\nu!}L_{n-\nu}^{(\beta)}(x,k) \qquad (4)$$

PROOF.

$$(1-t)^{-1-\alpha}\exp\left[\left(\frac{-x^k t}{(1-t)^k}\right)\right]=(1-t)^{-(\alpha-\beta)}(1-t)^{-1-\beta}\exp\left[\frac{-x^k t}{(1-t)^k}\right]$$

$$=\sum_{\nu=0}^{\infty}\frac{(\alpha-\beta)_\nu}{\nu!}t^\nu\times\sum_{n=0}^{\infty}L_n^{(\beta)}(x,k)t^n=\sum_{n,\nu=0}^{\infty}\frac{(\alpha-\beta)_\nu}{\nu!}t^{n+\nu}L_n^{(\beta)}(x,k).$$

Therefore,

$$\sum_{n=0}^{\infty}L_n(x,k)t^n=\sum_{n=0}^{\infty}\sum_{\nu=0}^{n}\frac{(\alpha-\beta)_\nu}{\nu!}L_{n-\nu}^{(\beta)}(x,k)t^n.$$

Equating coefficients of t^n on both sides, we obtain

$$L_n^{(\alpha)}(x,k)=\sum_{\nu=0}^{n}\frac{(\alpha-\beta)_\nu}{\nu!}L_{n-\nu}^{(\beta)}(x,k).$$

Putting $k=1$ and recalling that $L_n^{(\alpha)}(x,1)=L_n^{(\alpha)}(x)$, we have

$$L_n^{(\alpha)}(x)=\sum_{\nu=0}^{n}\frac{(\alpha-\beta)_\nu}{\nu!}L_{n-\nu}^{(\beta)}(x),$$

a known result [2; equation (2)].

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REFERENCES

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