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DENSIFYING MAPPINGS AND THEIR FIXED POINTS

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1. Introduction

Let A be a bounded subset of a metric space (X, d). Kuratowski [3] introduced the concept of $\alpha(A)$, the measure of non-compactness of A. $\alpha(A)$ denotes the infimum of all $\varepsilon > 0$ such that A admits a finite covering consisting of subsets with diameter $< \varepsilon$.

The following properties of α can be easily verified. For proofs, one can refer to Darbo [1] and Nussbaum [4].

(i) $0 \leqslant \alpha(A) \leqslant \delta(A)$ where $\delta(A)$ is the diameter of A,

(ii) $A \subset B \Rightarrow \alpha(A) \leqslant \alpha(B)$,

(iii) $\alpha(A) = \alpha(\overline{A})$ where $\overline{A}$ is the closure of A,

(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$,

(v) $\alpha(A) = 0 \Longleftrightarrow A$ is pre-compact (totally bounded). Further, if (X, d) is complete, $\alpha(A) = 0 = \alpha(\overline{A}) \Rightarrow \overline{A}$ is compact.

2. Furi and Vignoli [2] introduced the following two definitions.

DEFINITION 1. A continuous mapping T from a metric space (X, d) to itself is said to be *densifying* if for every bounded subset A of X with $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$.

Contractive mappings and completely continuous mappings are densifying.

DEFINITION 2. Let F be a real valued lower semi continuous function defined on $X \times X$. The mapping $T: X \rightarrow X$ is said to be *weakly F-contractive* if and only if

$$F(Tx, Ty) < F(x, y) \text{ for all } x, y \in X,\ x \neq y.$$

When F is the distance function d, we say T is *weakly contractive.*

They have proved the following:

THEOREM A. *Let T be a densifying and weakly F-contractive mapping defined from a complete metric space (X, d) to itself. If for some $x_0 \in X$, the sequence of iterates starting from x_0 is bounded, then T has a unique fixed point in X.*

3. We prove a similar result which yields a unique common fixed point for a pair of densifying mappings. We first have the following

LEMMA. *T_1 and T_2 are two densifying mappings from a metric space (X,d) to itself if and only if for every pair of bounded subsets A and B of X we have*

(3.1) $\alpha(T_1(A)\cup T_2(B))<\alpha(A\cup B)$ *whenever* $\alpha(A\cup B)>0$.

PROOF. Suppose condition (3.1) holds. Letting $B=\phi$ we obtain that $\alpha(T_1(A))<\alpha(A)$ whenever $\alpha(A)>0$, i.e. T_1 is densifying. Similarly T_2 is also densifying. Conversely, suppose T_1 and T_2 are densifying. Let $\alpha(A\cup B)>0$, i.e. Max $\{\alpha(A), \alpha(B)\}>0$. Three cases arise. If $\alpha(A)$ and $\alpha(B)$ are >0, then $\alpha(T_1(A))<\alpha(A)$ and $\alpha(T_2(B))<\alpha(B)$. Hence max $\{\alpha(T_1(A)), \alpha(T_2(B))\}<\max\{\alpha(A),\alpha(B)\}$, i.e. $\alpha(T_1(A)\cup T_2(B))<\alpha(A\cup B)$. If $\alpha(A)>0$ and $\alpha(B)=0$, then $\alpha(T_1(A))<\alpha(A)$; and $\alpha(B)=0$ implies B is totally bounded. The continuous image $T_2(B)$ is also totally bounded and $\alpha(T_2(B))=0$. Hence $\alpha(T_1(A)\cup T_2(B))=\max\ \{\alpha(T_1(A)), \alpha(T_2(B))\}=\alpha(T_1(A))<\alpha(A)\leqslant\alpha(A\cup B)$, i.e. $\alpha(T_1(A)\cup T_2(B))<\alpha(A\cup B)$. Similarly this result again follows if $\alpha(A)=0$ and $\alpha(B)>0$. Hence the lemma.

The following definition was introduced in [5].

DEFINITION 3. Let $S=\{T_1, T_2\}$ be a pair of self mappings of a metric space (X,d) into itself. For $x_0\in X$, the sequence $J_S(x_0)=\{x_0, T_1x_0, T_2T_1x_0, T_1T_2T_1x_0, \cdots\}$ is called *the joint sequence of iterates of S at x_0*.

THEOREM 1. *Let $S=\{T_1, T_2\}$ be a pair of commutative densifying mappings defined on a complete metric space (X,d) such that T_1T_2 is weakly F-contractive. If for some $x_0\in X$ the joint sequence of iterates $J_S(x_0)$ of S at x_0 is bounded, then T_1 and T_2 have a unique common fixed point in X.*

PROOF. Let $M=J_S(x_0)=\{x_0, T_1x_0, T_2T_1x_0, \cdots\}$. Denote $M_1=\{x_0, T_2T_1x_0, T_2T_1T_2T_1x_0, \cdots\}$ and $M_2=\{T_1x_0, T_1T_2T_1x_0, \cdots\}$. Such that $M=M_1\cup M_2$. Since $T_1(M_1)=M_2$ and $T_2(M_2)=M_1\setminus\{x_0\}$ $M=T_1(M_1)\cup T_2(M_2)\cup\{x_0\}$. Therefore $\alpha(M)=\alpha(T_1(M_1)\cup T_2(M_2)\cup\{x_0\})=\max\ \{\alpha(T_1(M_1)\cup T_2(M_2)), \alpha(\{x_0\})\}=\alpha(T_1(M_1)\cup T_2(M_2))$. If $\alpha(M)=\alpha(M_1\cup M_2)>0$ then we must have $\alpha(T_1(M_1)\cup T_2(M_2))<\alpha(M_1\cup M_2)$ which will give a contradiction. Hence $\alpha(M)=0$, and by property (v) $\overline{M}$ is compact. Consider the function $\phi:\overline{M}\longrightarrow R$ defined by $\phi(x)=F(x, T_1T_2x)$ T_1T_2 being the composition of two continuous functions is continuous; and F being lower semi continuous, ϕ will be lower semi-continuous on compact $\overline{M}$. So it has a minimum at some point $z \in \overline{M}$. Now, $\overline{M}$ is invariant under T_1T_2 for $T_1T_2(M) = T_1(T_2(\overline{M})) \subset T_1(\overline{T_2(M)}) \subset \overline{T_1(T_2(M))} \subset \overline{M}$ since M is invariant under T_1T_2 Hence $T_1T_2(z) \in \overline{M}$. If $z \neq T_1T_2(z)$, $\phi(T_1T_2(z)) = F(T_1T_2(z), T_1T_2T_1T_2(z)) < F(z, T_1T_2)z)) = \phi(z)$. This contradicts the definition of z; hence $z = T_1T_2(z)$. z is the unique fixed point of T_1T_2; for if w is another fixed point, $F(T_1T_2(z), T_1T_2(w)) < F(z, w)$, i.e. $F(z, w) < F(z, w)$ which is not possible. Further, $z = T_1T_2(z)$ implies $T_1(z) = T_1T_1T_2(z) = T_1T_1T_2(z)$, i.e. $T_1(z)$ is a fixed point of T_1T_2. By the uniqueness of z, $T_1z = z$. Similarly $z = T(z)$. Hence z is the unique common fixed point of T_1 and T_2. This proves the theorem.

COROLLARY (i). *Let $S = \{T_1, T_2\}$ be a pair of commutative densifying self mappings on a bounded complete metric space (X, d), such that T_1T_2 is weakly contractive. Then there exists a unique common fixed point for T_1 and T_2.*

COROLLARY (ii). *Let X be a bounded complete metric space and let $S = \{T_1, T_2\}$ be a pair of commutative, completely continuous self mappings of X such that T_1T_2 is weakly F-contractive. Then there exists a unique common fixed point for T_1 and T_2.*

REMARKS. (i) The theorem can be generalized by replacing T_1 and T_2 by T_1^p and T_2^q, for any two positive integers p and q. This is so, since the unique common fixed point of T_1^p and T_2^q will also be the unique common fixed point of T_1 and T_2. ([7])

(ii) For the validity of this theorem, the definition of weak F-contractivity for a mapping T may be modified in any way so as to yield $F(Tx, T^2x) < F(x, Tx)$. For example, we may like Singh [6] take

$$F(Tx, Ty) < \frac{1}{3}\{F(x, Tx) + F(y, Ty) + F(x, y)\}$$

(iii) The theorem still holds if we merely assume that T_1T_2 is iteratively weakly F-contractive at all points of X, i.e. for every $x \in X$, there exists a positive integer $n(x)$ such that

$$F((T_1T_2)^{n(x)}\, x,\ (T_1T_2)^{n(x)}\, y) < F(x, y)\ \ \forall x, y \in X,\ \ x \neq y.$$

This definition was introduced by Thomas [8].

4. In this last section we generalize the notion of densifying mappings and extend Theorem A.

DEFINITION 4. A mapping $T: X \longrightarrow X$ is said to be $(p: q_1, q_2, \cdots, q_m)$ densifying if for $A \subset X$.

(4.1) $\qquad T^p$ is continuous and

(4.2) $\qquad \alpha(T^p(A)) < \sum_{j=1}^{m} a_j \, \alpha(T^{q_j}(A))$

whenever $\sum_{j=1}^{m} a_j \, (T^{q_j}(A))$ is finite and >0, where $p, q_1, q_2, \cdots, q_m$ are all non-negative integers and the a_j 's are non-negative reals such that $\sum_{j=1}^{m} a_j = 1$.

THEOREM 2. *Let $T: (X, d) \longrightarrow (X, d)$ be a $(p: q_1, q_2, \cdots, q_m)$ densifying mapping defined on a complete metric space (X, d) such that T^p is weakly F-contractive. If for some $x_0 \in X$, the sequence of iterates $\{x_n\}$ is bounded, then T has a unique fixed point in X.*

PROOF. Let $A = \bigcup_{n=0}^{\infty} \{x_n\}$ where $x_n = T \, x_{n-1}$, $n = 1, 2, \cdots$. Now, $T^p(A)$ and $T^{q_j}(A)$ for $j = 1, 2, \cdots, m$, all differ from A only by a finite number of terms; hence

$$\alpha(A) = \alpha(T^p(A)) = \alpha(T^{q_j}(A)).$$

If $\sum_{j=1}^{m} a_j \, \alpha(T^{q_j})(A))$ is finite and >0 then by (4.2)

$$\alpha(A) < \alpha(A) \left\{ \sum_{j=1}^{m} a_j \right\} = \alpha(A)$$

which is not possible. So we must have $\sum_{j=1}^{m} a_j \, \alpha(T^{q_j}(A)) = 0$. This implies that each term in the summation is independently. Since all the a_j 's cannot be zero, we have $\alpha(T^{q_j}(A)) = 0$ at least one j, i.e. $\alpha(T^p(A)) = 0$.
Therefore $\overline{T^p(A)}$ is compact, since X is complete. Consider the real valued function $\phi: \overline{T^p(A)} \longrightarrow R$ defined by $\phi(x) = F(x, T^p x)$. ϕ being the composition of a continuous and a lower semi-continuous function is itself lower semi-continuous and attains a minimum at a point $z \in \overline{T^p(A)}$. The continuity of T^p gives $T^p(\overline{T^p(A)}) \subset \overline{T^p(T^p(A))} = \overline{T^{2p}(A)} \subset \overline{T^p(A)}$, i.e. $T^p(z) \in \overline{T^p(A)}$. If $z \neq T^p(z)$, $\phi(T^p(z)) = F(T^p(z), \; T^{2p}(z)) < F(z, T^p(z)) = \phi(z)$. This contradicts the definition of z; hence $z = T^p(z)$. The weak F-contractivity of T^p immediately gives that z is the

unique fixed point of T^p. Also $T(z)=T(T^p(z))=T^p(Tz)$. Hence $z=T(z)$ by the uniqueness of z, i.e. z is the unique fixed point of T. Thus proves the theorem.

REMARKS. (i) If T is a $(p:q)$ densifying mapping, condition (4.2) would reduce to $\alpha(T^p(A))<\alpha(T^q(A))$.

(ii) If T is a $(p:0)$ densifying mapping then we have $\alpha(T^p(A))<\alpha(A)$, i.e. T^p is densifying. (see [8])

(iii) If T is a $(1:0)$ densifying mapping, T will be densifying, and Theorem 2 will reduce to Theorem A.

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