Kyungpook Math. J.
Volume 18, Number 2
December, 1978

T_0-IDENTIFICATION SPACES AND R_1 SPACES

By Charles Dorsett

1. Introduction

The R_1 axiom was introduced in 1961 by A.S. Davis [1]. In 1975 William Dunham [2] further investigated the R_1 axiom and proved that R_1 spaces, which are weaker than T_2 and regular spaces, preserve many of the properties of T_2 and regular spaces. In this paper T_0-identification spaces are further investigated, induced maps are introduced and investigated, and the results are used to obtain additional properties of R_1 spaces.

Listed below are definitions and theorems that will be utilized in this paper.

DEFINITION 1.1. A space (X, T) is R_1 iff for $x, y \in X$ such that $\overline{\{x\}} \neq \overline{\{y\}}$, there exist disjoint open sets U and V such that $\overline{\{x\}} \subset U$ and $\overline{\{y\}} \subset V$ [1].

THEOREM 1.1. *A space (X, T) is T_2 iff it is T_1 and R_1* [1].

DEFINITION 1.2. Let R be the equivalence relation on the space (X, T) defined by xRy iff $\overline{\{x\}} = \overline{\{y\}}$. Then the *$T_0$-identification space* of (X, Y) is $(X_0, Q(T))$, where X_0 is the set of equivalence classes of R and $Q(T)$ is the decomposition topology on X_0, which is T_0. Let $P_X : (X, T) \rightarrow (X_0, Q(T))$ be the natural map [3].

THEOREM 1.2. *A space (X, T) is R_1 iff $(X_0, Q(T))$ is T_2* [2].

THEOREM 1.3. *If (X, T) is locally compact R_1, then (X, T) is completely regular* [2]

2. T_0-identification spaces and induced maps

THEOREM 2.1. *The natural map $P_X : (X, T) \rightarrow (X_0, Q(T))$ is continuous, closed, open, onto, $P_X^{-1}(P_X(O)) = O$ for all $O \in T$, $P_X^{-1}(\mathrm{Fr}(\mathscr{O})) = \mathrm{Fr}(P_X^{-1}(\mathscr{O}))$ for all $\mathscr{O} \in Q(T)$, and $P_X^{-1}(\mathscr{C})$ is compact or connected iff $\mathscr{C}$ is compact or connected, respectively.*

PROOF. For each $z \in X$, let C_z be the equivalence class of R containing z.

Let $O\in T$ and let $x\in P_X^{-1}(P_X(O))$. Then there exists $y\in O$ such that $C_x=C_y$. Thus $\overline{\{x\}}=\overline{\{y\}}$ and $O\in T$ such that $y\in O$, which implies $x\in O$. Hence, $P_X^{-1}(P_X(O))\subset O$, which implies $O=P_X^{-1}(P_X(O))$. If $O\in T$, then $P_X^{-1}(P_X(O))=O\in T$, which implies $P_X(O)\in Q(T)$ and P_X is open. If C is closed, then $P_X(C)=X_0\backslash P_X(X\backslash C)$, which is closed, which implies P_X is closed. Since P_X is closed and $P_X^{-1}(P_X(C))=C$ for all closed subsets C of X, then $\overline{P_X^{-1}(\mathscr{O})}=P_X^{-1}(\overline{\mathscr{O}})$ for all $\mathscr{O}\in Q(T)$. If $\mathscr{O}\in Q(T)$, then $P_X^{-1}(\mathscr{O})\cup\mathrm{Fr}(P_X^{-1}(\mathscr{O}))=P_X^{-1}(\mathscr{O})\cup P_X^{-1}(\mathrm{Fr}(\mathscr{O}))$, which implies $\mathrm{Fr}(P_X^{-1}(\mathscr{O}))=P_X^{-1}(\mathrm{Fr}(\mathscr{O}))$. Since P_X is continuous, onto, and $P_X^{-1}(P_X(O))=O$ for all $O\in T$, then $P_X^{-1}(\mathscr{C})$ is compact iff $\mathscr{C}$ is compact and since P_X is continuous, onto, closed, and $P_X^{-1}(P_X(C))=C$ for all closed subsets C of X, then $P_X^{-1}(\mathscr{C})$ is connected iff $\mathscr{C}$ is connected.

THEOREM 2.2. *If f is a continuous function from the space (X,T) onto the space (Y,S), then the relation $f^*=\{(C_x,C_{f(x)})\mid C_x\in X_0\}$ is a continuous function from $(X_0,Q(T))$ onto $(Y_0,Q(S))$.*

PROOF. Let $C_{x_1},C_{x_2}\in X_0$ such that $C_{x_1}=C_{x_2}$. Then $C_{f(x_1)}=C_{f(x_2)}$, for suppose not. Then $\overline{\{f(x_1)\}}\neq\overline{\{f(x_2)\}}$, which implies $f(x_1)\notin\overline{\{f(x_2)\}}$ or $f(x_2)\notin\overline{\{f(x_1)\}}$, say $f(x_1)\notin\overline{\{f(x_2)\}}$. Then $f(x_1)\in Y\backslash\overline{\{f(x_2)\}}$, $x_1\in f^{-1}(Y\backslash\overline{\{f(x_2)\}})\in T$, and $x_2\notin f^{-1}(Y\backslash\overline{\{f(x_2)\}})$. Since $C_{x_1}=C_{x_2}$, then $\overline{\{x_1\}}=\overline{\{x_2\}}$ and since $x_1\in f^{-1}(Y\backslash\overline{\{f(x_2)\}})\in T$, then $x_2\in f^{-1}(Y\backslash\overline{\{f(x_2)\}})$, which is a contradiction. Hence f^* is a function. Since f is onto, then f^* is onto and since P_X is open, f and P_Y are continuous, and $(f^*)^{-1}(\mathscr{O})=P_X(f^{-1}(P_Y^{-1}(\mathscr{O})))$ for all $\mathscr{O}\in Q(S)$, then f^* is continuous.

DEFINITION 2.1. If f is a continuous function from a space (X,T) onto a space (Y,S), then the function $f^*:(X_0,Q(T))\to(Y_0,Q(S))$ defined by $f^*(C_x)=C_{f(x)}$ is the *induced map from $(X_0,Q(T))$ onto $(Y_0,Q(S))$ determinded by f.*

THEOREM 2.3. *Let f be a continuous function from (X,T) onto (Y,S). If f is open (closed), then f^* is open (closed).*

PROOF. Suppose f is open. Since P_Y and f are open, then $f^*(\mathscr{O})=P_Y(f(P_X^{-1}(\mathscr{O})))\in Q(S)$ for all $\mathscr{O}\in Q(T)$, which implies f^* is open.

Suppose f is closed. Since P_Y and f are closed, then $f^*(\mathscr{C})=P_Y(f(P_X^{-1}(\mathscr{C})))$ is closed for all closed subsets $\mathscr{C}$ of X_0, which implies f^* is closed.

The converse of Theorem 2.3 is false even if f is 1-1.

3. R_1 spaces

THEOREM 3.1. *A space (X, T) is T_2 iff (X, T) is R_1 and every topology on X finer than T is R_1.*

PROOF. Suppose (X, T) is T_2. Then every topology on X finer than T is T_2 and since every T_2 space is R_1, then (X, T) is R_1 and every topology on X finer than T is R_1.

Conversely, suppose (X, T) is R_1 and every topology on X finer than T is R_1. Then (X, T) is T_1, for suppose not. Then there exists $x\in X$ such that $\overline{\{x\}}\neq\{x\}$. Let $S=T\cup\{O\cup\{x\}\mid O\in T\}$. Then S is a topology on X and $T\subset S$, which implies (X, S) is R_1. Let y be a point in the T-closure of $\{x\}$ such that $x\neq y$. Then $x, y\in X$ such that $\overline{\{x\}}_{(X,S)}\neq\overline{\{y\}}_{(X,S)}$, which implies there exist $U, V\in S$ such that $\overline{\{x\}}_{(X,S)}\subset U$, $\overline{\{y\}}_{(X,S)}\subset V$, and $U\cap V=\phi$. Since $V\in S$ and $x\notin V$, then $V\in T$. Thus there exists $V\in T$ such that $y\in V$ and $x\notin V$, which contradicts $y\in\overline{\{x\}}_{(X,T)}$. Hence, (X, T) is R_1 and T_1, which implies (X, T) is T_2.

THEOREM 3.2. *Let (X, T) be compact R_1 and let $x\in X$. Then the component A_x of X containing x equals the quasicomponent B_x of X containing x.*

PROOF. Since (X, T) is compact R_1, then $(X_0, Q(T))$ is compact T_2. Let $\mathscr{A}_{C_x}$ and $\mathscr{B}_{C_x}$ denote the component and quasicomponent of X_0 containing C_x, respectively. Then $\mathscr{A}_{C_x}=\mathscr{B}_{C_x}$. Since $\{\mathscr{O}\subset X_0\mid C_x\in\mathscr{O}$ and $\mathscr{O}$ is closed open in $X_0\}=\{P_X(O)\mid x\in O$ and O is closed open in $X\}$, then $\mathscr{B}_{C_x}=\cap\{P_X(O)\mid x\in O$ and O is closed open in $X\}=P_X(\cap\{O\subset X\mid x\in O$ and O is closed open in $X\})=P_X(B_x)$ and since $\mathscr{B}_{C_x}$ is connected and B_x is closed, then $B_x=P_X^{-1}(\mathscr{B}_{C_x})$, which is connected. Hence $B_x\subset A_x$, which implies $B_x=A_x$.

THEOREM 3.3. *Let (X, T) be compact R_1, let D be a closed subset of X, let C be a component of D, and let U be open such that $C\subset U$. Then there exists an open set V such that $C\subset V\subset U$ and* $\mathrm{Fr}(V)\cap D=\phi$.

PROOF. Since $P_X(D)$ is a closed subset of the compact T_2 space $(X_0, Q(T))$,

$P_X(C)$ is a component of $P_X(D)$, and $P_X(U)$ is open in X_0 such that $P_X(C)\subset P_X(U)$, then there exists an open set $\mathscr{V}$ in X_0 such that $P_X(C)\subset\mathscr{V}\subset P_X(U)$ and $\mathrm{Fr}(\mathscr{V})\cap P_X(D)=\phi$. Thus $P_X^{-1}(\mathscr{V})$ is open in X, $C\subset P_X^{-1}(\mathscr{V})\subset U$, and $\mathrm{Fr}\,(P_X^{-1}(\mathscr{V}))\cap D=P_X^{-1}(\mathrm{Fr}(\mathscr{V}))\cap D=\phi$.

The next two theorems can be proven by an argument similar to that for Theorem 3.3.

THEOREM 3.4. *If (X,T) is compact connected R_1 and C is a component of $\overline{G}$, where $G\in T$ and $\overline{G}\subsetneq X$, then $C\cap\mathrm{Fr}(G)\neq\phi$.*

THEOREM 3.5. *If (X,T) is compact connected R_1 and C is a proper closed connected subset of X, then there exists a closed connected subset K of X such that $C\subsetneq K\subsetneq X$.*

THEOREM 3.6. *Let (X,T) be locally compact R_1, let D be a closed subset of X with compact component C, and let $U\in T$ such that $C\subset U$. Then there exists $V\in T$ such that $\overline{V}$ is compact, $C\subset V\subset\overline{V}\subset U$, and* $\mathrm{Fr}(V)\cap D=\phi$.

PROOF. By Theorem 1.3 (X,T) is regular. For each $x\in C$, let $W_x\in T$ such that $\overline{W}_x$ is compact and $x\in W_x\subset\overline{W}_x\subset U$. Then $\{W_x\}_{x\in C}$ is an open cover of C and there exists a finite subcover $\{W_{x_i}\}_{i=1}^n$. Let $Y=\bigcup_{i=1}^n\overline{W}_{x_i}$. Then C is a component of $D\cap Y$, which is a closed subset of the compact R_1 space (Y,T_Y), and $C\subset\bigcup_{i=1}^n W_{x_i}\in T_Y$. Thus by Theorem 3.3, there exists $V\in T_Y$ such that $C\subset V\subset\bigcup_{i=1}^n W_{x_i}$ and $\mathrm{Fr}_Y(V)\cap(D\cap Y)=\phi$. Then $C\subset V\in T$, $\overline{V}$ is compact, and $\mathrm{Fr}(V)\cap D=\phi$.

THEOREM 3.7. *Let (X,T) be R_1, let A be a closed compact subset of X, let C be a component of A, and let $U\in T$ such that $C\subset U$. Then there exists $W\in T$ such that $C\subset W\subset U$ and* $\mathrm{Fr}(W)\cap A=\phi$.

PROOF. Since A is a closed subset of the compact R_1 space (A,T_A), C is a component of A, and $C\subset U\cap A\in T_A$, then by Theorem 3.3, there exists $V\in T_A$ such that $C\subset V\subset U\cap A$ and $\mathrm{Fr}_A(V)\cap A=\phi$. Then V is closed open in A and $V=O\cap A$, where $O\in T$ and $O\subset U$. If $\mathrm{Fr}(O)\cap A=\phi$, then $O\in T$ such that $C\subset O\subset U$ and $\mathrm{Fr}(O)\cap A=\phi$. Thus consider the case that $\mathrm{Fr}(O)\cap A\neq\phi$. If $x\in\mathrm{Fr}(O)\cap A$, then $x\notin V$ and since (X,T) is R_1 and V is closed compact, then there exists $O_x\in T$ such that $x\in O_x\subset\bar{O}_x\subset X\setminus V$. For each $x\in \mathrm{Fr}(O)\cap A$, let $O_x\in T$ such that $x\in O_x\subset\bar{O}_x\subset X\setminus V$. Then $\{O_x\}_{x\in \mathrm{Fr}(O)\cap A}$ is an open cover of $\mathrm{Fr}(O)\cap A$ and there exists a finite subcover $\{O_{x_i}\}_{i=1}^{n}$. Let $W=O\setminus\bigcup_{i=1}^{n}\bar{O}_{x_i}\in T$. Then $C\subset W\subset U$ and $\mathrm{Fr}(W)\cap A=\phi$.

THEOREM 3.8. *If $(\mathscr{D},Q)$ is an upper semi-continuous decomposition of a locally compact R_1 space (X,T) into closed subsets of X with compact components, $\mathscr{G}=\{C\mid C$ is a component of $D, D\in\mathscr{D}\}$, and Q_1 is the decomposition topology on $\mathscr{G}$, then $(\mathscr{G},Q_1)$ is an upper semicontinuous decomposition of X.*

PROOF. Let $C\in\mathscr{G}$ and let $U\in T$ such that $C\subset U$. Then there exists $D\in\mathscr{D}$ such that C is a component of D. By Theorem 3.6, there exists $V\in T$ such that $C\subset V\subset U$ and $\mathrm{Fr}(V)\cap D=\phi$. Then $D\in\mathscr{D}$ and $X\setminus\mathrm{Fr}(V)\in T$ such that $D\in X\setminus\mathrm{Fr}(V)$, which implies there exists $W\in T$ such that $D\subset W\subset X\setminus\mathrm{Fr}(V)$ and such that if $D_1\in\mathscr{D}$ and $D_1\cap W\neq\phi$, then $D_1\subset X\setminus\mathrm{Fr}(V)$. Then $V\cap W\in T$ such that $C\subset V\cap W\subset U$ and such that if $B_1\in\mathscr{G}$ and $B_1\cap(V\cap W)\neq\phi$, then $B_1\subset V\subset U$. Hence, $(\mathscr{G},Q_1)$ is an upper semi-continuous decomposition of (X,T).

The next theorem follows from Theorem 3.7 and an argument similar to that for Theorem 3.8.

THEOREM 3.9. *If $(\mathscr{D},Q)$ is an upper semi-continuous decomposition of an R_1 space (X,T) into closed compact subset of X, $\mathscr{G}=\{C\mid C$ is a component of D, $D\in\mathscr{D}\}$, and Q_1 is the decomposition topology on $\mathscr{G}$, then $(\mathscr{G},Q_1)$ is an upper semicontinuous decomposition of (X,T).*

4. Continuous images of R_1 spaces

Example 13.9 (b) in Willard's book [3] show that if f is a continuous open function from a R_1 space (X,T) onto a space (Y,S) such that $f^{-1}(y)$ is closed compact for all $y\in Y$, then (Y,S) need not be R_1.

THEOREM 4.1. *If f is a continuous open function from (X,T) onto $(Y.S)$, then (Y,S) is R_1 iff $\{(x_1,x_2)\mid\overline{\{f(x_1)\}}=\overline{\{f(x_2)\}}\}$ is a closed subset of $X\times X$.*

PROOF. Suppose (Y,S) is R_1. Then f^* is a continuous function from $(X_0, Q(T))$ onto the T_2 space $(Y_0,Q(S))$, which implies $\mathscr{C}=\{(C_{x_1},C_{x_2})\mid f^*(C_{x_1})=f^*$

$(C_{x_2})\}$ is closed in $X_0 \times X_0$. Since $\mathscr{C}$ is closed in $X_0 \times X_0$, then $\{(x_1, x_2) \mid \overline{\{f(x_1)\}} = \overline{\{f(x_2)\}}\}$ is closed in $X \times X$.

Conversely, suppose $\{(x_1, x_2) \mid \overline{\{f(x_1)\}} = \overline{\{f(x_2)\}}\}$ is a closed subset of $X \times X$. Then $\mathscr{C} = \{(C_{x_1}, C_{x_2}) \mid f^*(C_{x_1}) = f^*(C_{x_2})\}$ is closed in $X_0 \times X_0$ and since f^* is a continuous open function from $(X_0, Q(T))$ onto $(Y_0, Q(S))$ such that $\mathscr{C}$ is closed in $X_0 \times X_0$, then $(Y_0, Q(S))$ is T_2, which implies (Y, S) is R_1.

Example 13.9 (a) of Willard's book [3] shows that the continuous closed image of an R_1 space need not be R_1.

THEOREM 4.2. *If f is a continuous closed function from an R_1 space (X, T) onto a space (Y, S) such that $f^{-1}(y)$ is compact for all $y \in Y$, then (Y, S) is R_1.*

PROOF. Let $\mathscr{D} = \{f^{-1}(y) \mid y \in Y\}$ and let Q be the decomposition topology on $\mathscr{D}$. Then $(\mathscr{D}, Q)$ is an upper semicontinuous decomposition of X into compact sets. Let P be a natural map from (X, T) onto $(\mathscr{D}, Q)$. Let $C_1, C_2 \in \mathscr{D}$ such that $\overline{\{C_1\}} \neq \overline{\{C_2\}}$. Then $C_1 \notin \overline{\{C_2\}}$ or $C_2 \notin \overline{\{C_1\}}$, say $C_1 \notin \overline{\{C_2\}}$. Let $x \in C_1$. If $y \in C_2$, then $\overline{\{x\}} \neq \overline{\{y\}}$ and there exist disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. For each $y \in C_2$, let U_y and V_y be disjoint open sets such that $x \in U_y$ and $y \in V_y$. Then $\{V_y\}_{y \in C_2}$ is an open cover of C_2 and there exists a finite subcover $\{V_{y_i}\}_{i=1}^{n}$. Thus

$$x \in \bigcap_{i=1}^{n} U_{y_i} \in T,\ C_2 \subset \bigcup_{i=1}^{n} V_{y_i} \in T,\ \text{and}\ (\bigcap_{i=1}^{n} U_{y_i}) \cap (\bigcup_{i=1}^{n} V_{y_i}) = \phi.$$

For each $x \in C_1$, let U_x and V_x be disjoint open sets such that $x \in U_x$ and $C_2 \subset V_x$. Then $\{U_x\}_{x \in C_1}$ is an open cover of C_1 and there exists a finite subcover $\{U_{x_i}\}_{i=1}^{m}$. Thus,

$$C_1 \subset \bigcup_{i=1}^{m} U_{x_i} = U_1 \in T,\ C_2 \subset \bigcap_{i=1}^{m} V_{x_i} = U_2 \in T,\ \text{and}\ U_1 \cap U_2 = \phi.$$

Then $C_1 \in \mathscr{U}_1 = \{D \in \mathscr{D} \mid D \subset U_1\} \in Q$, $C_2 \in \mathscr{U}_2 = \{D \in \mathscr{D} \mid D \subset U_2\} \in Q$, and $\mathscr{U}_1 \cap \mathscr{U}_2 = \phi$. If $z \in C_1$, then $z \in P^{-1}(\mathscr{U}_1) \in T$, which implies $\overline{\{z\}} \subset P^{-1}(\mathscr{U}_1)$, and since P is closed, then $\overline{\{C_1\}} \subset P(\overline{\{z\}}) \subset \mathscr{U}_1$. By a similar argument, $\overline{\{C_2\}} \subset \mathscr{U}_2$. Hence, $(\mathscr{D}, Q)$ is R_1, which implies (Y, S) is R_1.

THEOREM 4.3. *If f is a continuous closed function from an R_1 space (X, T) onto a space (Y, S) such that $f^{-1}(y)$ is closed compact for all $y \in Y$, then (Y, S)*

is T_2.

PROOF. Let $\mathscr{D}=\{f^{-1}(y)\mid y\in Y\}$ and let Q be the decomposition topology on $\mathscr{D}$. Then $(\mathscr{D},Q)$ is an R_1 upper semicontinuous decomposition of X into closed sets. If $C_1, C_2\in\mathscr{D}$ such that $C_1\neq C_2$, then $C_1\in\mathscr{U}=\{D\in\mathscr{D}\mid D\subset X\backslash C_2\}\in Q$ and $C_2\notin\mathscr{U}$, which implies $(\mathscr{D},Q)$ is T_1. Hence, $(\mathscr{D},Q)$ is T_2, which implies (Y,S) is T_2.

THEOREM 4.4. *If f is a continuous function from a compact R_1 space (X,T) onto a T_0 space (Y,S), then (Y,S) is T_2 iff f is closed.*

PROOF. Suppose f is closed. If $O\in S, y\in O$, and $y=f(x)$, then $x\in f^{-1}(O)\in T$, which implies $\overline{\{x\}}\subset f^{-1}(O)$ and $\overline{\{y\}}\subset f(\overline{\{x\}})\subset O$. If $y\in Y$, then $\overline{\{y\}}=\{y\}$, for suppose not. Then there exists $y\in Y$ such that $\{y\}\subsetneq\overline{\{y\}}$. Let $z\in\overline{\{y\}}\backslash\{y\}$. Since (Y,S) is T_0, then $y\notin\overline{\{z\}}$. Thus $y\in Y\backslash\overline{\{z\}}\in S$, which implies $\overline{\{y\}}\subset Y\backslash\{z\}$, which is a contradiction. Hence, (Y,S) is T_1 and $f^{-1}(y)$ is compact closed for all $y\in Y$, which implies (Y,S) is T_2.

The straightforward proof of the converse is omitted.

THEOREM 4.5. *If f is a continuous closed open function from a regular space (X,T) onto a space (Y,S), then (Y,S) is R_1.*

PROOF. Since f is continuous closed open, then f^* is continuous closed open; since (X,T) is regular, then (X,T) is R_1 and $(X_0,Q(T))$ is T_2; and since $(X_0, Q(T))$ is an upper semi-continuous decomposition of the regular space (X,T) into compact sets, then $(X_0,Q(T))$ is regular. Thus f^* is a continuous closed open function from the T_3 space $(X_0,Q(T))$ onto $(Y_0,Q(S))$, which implies $(Y_0,Q(S))$ is T_2. Hence, (Y,S) is R_1.

THEOREM 4.6. *If f is a continuous closed function from a compact R_1 space (X,T) onto a T_0 space (Y,S), then $f=g\circ h$, where h is monotone and g is light.*

PROOF. By Theorem 4.4. (Y,S) is T_2. Let $\mathscr{D}=\{f^{-1}(y)\mid y\in Y\}$, let Q be the decomposition topology on $\mathscr{D}$, let $\mathscr{G}=\{C\mid C$ is a component of $D, D\in\mathscr{D}\}$, and let Q_1 be the decomposition topology on $\mathscr{G}$. By Theorem 3.9 $(\mathscr{G},Q_1)$ is an upper semi-continuous decomposition of X. Let h be the natural map from (X,T) onto $(\mathscr{G},Q_1)$. Then h is closed monotone and by Theorem 4.3 $(\mathscr{G},Q_1)$ is compact T_2. Let $g:(\mathscr{G},Q_1)\rightarrow(Y,S)$ defined by $g(C)=y$ iff C is a component of $f^{-1}(y)$. Since $h^{-1}(g^{-1}(O))=f^{-1}(O)\in T$ for all $O\in S$, then g is continuous. Let $y\in Y$, let $C_1\in g^{-1}(y)$, and let $C_2\in g^{-1}(y)$ such that $C_2\neq C_1$. Then there exist disjoint open sets U_1 and U_2 such that $C_1\subset U_1$ and $C_2\subset U_2$. By Theorem 3.3 there exists $V\in T$ such that $C_1\subset V\subset U_1$ and $\mathrm{Fr}(V)\cap f^{-1}(y)=\phi$. Then $C_1\in \mathscr{V}=g^{-1}(y)\cap\{C\in\mathscr{G}\mid C\subset V\}=g^{-1}(y)\cap h(\bar{V})$, which is closed open in $g^{-1}(y)$, and $C_2\notin\mathscr{V}$. Thus the component of $g^{-1}(y)$ containing C_1 is $\{C_1\}$. Hence, $g^{-1}(y)$ is totally disconnected for each $y\in Y$, which implies g is light.

Louisiana Tech. University
Ruston, Louisiana 71272
U.S.A.

REFERENCES

[1] Davis, A., *Indexed Systems of Neighborhoods for General Topological Spaces*, The American Mathematical Monthly, 68(1961), 886—893.

[2] Dunham, W., *Weakly Hausdorff Spaces*, Kyungpook Mathematical Journal, 15(1975), 41—50.

[3] Willard S., *General Topology*, Addison-Wesley Publishing Company, 1970.