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ON A GENERALIZATION OF A THEOREM OF P. HILL

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The object of this short note is to generalize a well known Theorem of P. Hill which he proves for primary groups [2 ; Theorem 2.4] for the case of a module over bounded hereditary noetherian prime ring. The technique of the proof is entirely new and much easier than the given one by Paul Hill.

We recall that if M is a module over a ring R then a submodule N of M is called pure if $rN=N\cap rM$ for every $r\in R$. The following lemma is seemingly known.

LEMMA 1. *If M is a module over a ring R and N_1, N_2 are submodules of M then followings hold.*

(1) *If $N_1\cap N_2$ is pure in N_2 then N_1 is pure in N_1+N_2.*

(2) *If N_1+N_2 is pure in M and $N_1\cap N_2$ is pure in N_2 then N_1 is pure in M.*

(3) *If N_1+N_2 is pure in M and $N_1\cap N_2$ is pure in M then N_1 and N_2 are pure in M.*

(4) *If $N_1\cap N_2$ is pure in N_1+N_2 then N_1 and N_2 are pure in N_1+N_2.*

PROOF. For any $r\in R$,

$$N_1\cap r(N_1+N_2)=N_1\cap(rN_1+rN_2)$$
$$=rN_1+(N_1\cap rN_2)\,;$$

but as $N_1\cap N_2$ is pure in N_2, it is easy to see that $N_1\cap rN_2\subset rN_1$. Hence $N_1\cup r(N_1+N_2)=rN_1$ and we get N_1 to be pure submodule of N_1+N_2.

Now using (1) and transitive property of purity, (2), (3), (4) trivially follow.

Now if M is a module over a bounded hereditary noetherian prime ring R, then the following has been proved by Singh [3 ; Corollary 4].

THEOREM 2. *If M is a divisible module over a bounded (hnp)-ring R then M is injective.*

As defined in [1], a submodule N of an arbitrary module M over a ring R is called dense if M/N is divisible.

Now the following Theorem generalizes [2 ; Theorem 2.4.].

THEOREM 3. *If M is a module over a bounded (hnp)-ring R. If N is submodule of M and U is a pure and dense submodule of N, then there exists a pure and dense submodule V of M such that $U=V\cap N$.*

PROOF. Appealing to Theorem 2, $M/U=N/U\oplus V/U$ for some submodule V/U of M/U. Trivially $U=N\cap V$ and V is dense in M and $M=N+V$. Now appealing to lemma 1 we get V to be pure submodule of M. Hence the Theorem.

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